No lecture on 18 October.

Recent development, new approach introduced in 2012.

I. Introduction

Thm. (Faltings, 1983, formally known as the Mordell conjecture)

$K/\mathbb{Q}$ finite, X/K smooth connected projective curve, $g(X) \geq 2 \Rightarrow X(K)$ is finite.

Thm. (Faltings, 1983, Shafarevich conjecture)

$S \subseteq \Sigma_{K,f}$ finite set of places, $g \geq 1$, $d \geq 1$

Then up to iso there are only fin many pairs (A, λ) where

- A/K abelian variety of dim g, good reduction away from S
- $\lambda : A \longrightarrow A^\vee$ polarisation of deg d.

Here Σ_K denotes the set of places, $\Sigma_{K,f}$ the finite places, $\Sigma_{K,\infty}$ the infinite ones.

§1 Height

$K/\mathbb{Q}$ fixed finite

For $v \in \Sigma_K$, normalise $|\cdot|_v$:

$$\begin{cases} |\pi_v|_v = q_v^{-1} \text{ where } \pi_v \in \mathcal{O}_v \text{ uniformiser, } q_v = \# \mathcal{O}_v/\pi_v \mathcal{O}_v & \text{if } v \text{ non-arch} \\ |\cdot|_v \text{ usual abs. value} & \text{if } K_v \cong \mathbb{R} \text{ or } \mathbb{C} \end{cases}$$

Thm. (Product formula)

$$\forall f \in K^\times : \prod_{v \in \Sigma_K} |f|_v^{\varepsilon_v} = 1 \quad \text{where} \quad \varepsilon_v = \begin{cases} 2 & K_v \cong \mathbb{R} \text{ or } \mathbb{C} \\ 1 & \text{o/w} \end{cases}$$

Def. $H_K : \mathbb{P}^n(K) \longrightarrow \mathbb{R}_{>0}$

$$(x_0, \dots, x_n) \longmapsto \left(\prod_{v \in \Sigma_K} \|(x_0, \dots, x_n)\|_v \right)^{1/[K:\mathbb{Q}]}$$

with $$\|(x_0, \dots, x_n)\|_v = \begin{cases} \sup \{ |x_i|_v \mid i = 0, \dots, n \} & v \text{ non-arch.} \\ \left(\sum_{i=0}^n |x_i|_v^2 \right)^{\varepsilon_v/2} & v \text{ archi.} \end{cases}$$

Product formula $\Rightarrow H_K$ is well-def'd

If $K_1 \subseteq K_2 \Rightarrow H_{K_2}|_{\mathbb{P}^n(K_1)} = H_{K_1}$

Def. height on $\mathbb{P}^n(\overline{\mathbb{Q}})$. $h = \log H : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$

Thm. (Northcott property) Given $C>0$, $\#\{x\in\mathbb{P}^n(K) \mid h(x)<C\}<\infty$.

For $d\geq 1$, we even have $\#\{x\in\mathbb{P}^n(\overline{\mathbb{Q}}) \mid h(x)<C,\ [\mathbb{Q}(x):\mathbb{Q}]<d\}<\infty$

Construction. X/K proper variety, $\mathcal{L}/X$ line bundle, $s_0,\dots,s_n\in\Gamma(X,\mathcal{L})$ global generators. $\varphi:=\varphi|_{\mathcal{L},s_0,\dots,s_n}: X\longrightarrow\mathbb{P}^n_K$

$$\begin{aligned} h_{\mathcal{L},s}: X(\overline{\mathbb{Q}}) &\longrightarrow \mathbb{R} \\ x &\longmapsto h(\varphi(x)) \end{aligned}$$

- Weil: up to scalar and bounded factor, this only depends on $\mathcal{L}$.
- If $\mathcal{L}$ is ample: $\forall C$ $\#\{x\in X(\overline{\mathbb{Q}}) \mid h_{\mathcal{L},s_0,\dots,s_n}(x)<C\}<\infty$ by Northcott

§ 2 Arakelov heights

Arakelov '74, Faltings

For an introduction: Faltings / Wüstholz : Rational points

Def. Metrised line bundle on $S=\operatorname{Spec}\mathcal{O}_K$ is a pair $(\mathcal{L},\|\cdot\|)$ where

- $\mathcal{L}$ a rank 1 $\mathcal{O}_K$-module
- $\|\cdot\|=(\|\cdot\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$ where $\|\cdot\|_\sigma:\mathcal{L}\otimes_{\mathcal{O}_K,\sigma}\mathbb{C}\longrightarrow\mathbb{R}_{\geq 0}$ is an hermitian metric, i.e. has the form $\|x\|_\sigma=\sqrt{(x,x)_\sigma}$ for some $(\cdot,\cdot)_\sigma:\mathcal{L}_\sigma\times\mathcal{L}_\sigma\longrightarrow\mathbb{C}$ positive definite and hermitian.

s.t. $\forall s\in\mathcal{L}$ $\forall\sigma$: $\|F_\infty(\sigma(s))\|_{F_\infty\sigma}=\|\sigma(s)\|_\sigma$. Here F_∞ denotes the complex conjugation.

Then $(\widehat{\operatorname{Pic}}(S),\otimes):=$ group of iso classes of metrised lbs

Def. Arakelov degree $\widehat{\deg}:\widehat{\operatorname{Pic}}(S)\longrightarrow\mathbb{R}$

$$(\mathcal{L},\|\cdot\|)\longmapsto\log\#\mathcal{L}/s\mathcal{L}-\sum_{v\in\Sigma_{K,\infty}}\varepsilon_v\|s\|_v$$

where $s\in\mathcal{L}\setminus\{0\}$. By prod. formula, we have independence of s.

Easy to check: $\widehat{\deg}$ is a gp. homomorphism.

Rmk. $X\longrightarrow S$ projective, then $X(K)=X(S)$ by valuative criterion of properness

Def. $X\longrightarrow S$ projective flat

1) Metrised line bundle on X: $(\mathcal{L},\|\cdot\|)$ s.t.

- $\mathcal{L}$ lb on X
- $\|\cdot\|=(\|\cdot\|_\sigma)_\sigma$, $\|\cdot\|_\sigma$ smooth herm. metric on $\mathcal{L}_\sigma$ over $(X\times_{\mathcal{O}_K,\sigma}\mathbb{C})(\mathbb{C})$

s.t. $F_\infty^*\|\cdot\|_\sigma=\|\cdot\|_{F_\infty,\sigma}$

2) $\hat{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$ metrized lb on X.

The Arakelov height wrt $\hat{\mathcal{L}}$ is $h_{\hat{\mathcal{L}}}: X(K) \longrightarrow \mathbb{R}$, $x \longmapsto \widehat{\deg}(\hat{\mathcal{L}}|_{\overline{\{x\}}})$

E.g. recover h on $\mathbb{P}^n(K)$ for $\hat{\mathcal{L}} = (\mathcal{O}(1), \|\cdot\|_{std})$ up to factor $[K:\mathbb{Q}]$.

§3 Sketch of Faltings' proof for the Sh. Conj

- Consider $X := A_{g,d} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ coarse moduli of (A,λ), $\dim A = g$, $\deg \lambda = d$
- Have "canonical" metrised ample lb $\hat{\mathcal{M}}$ on $A_{g,d}$, descended from some power $\left(\Lambda^g e^* \Omega^1_{\mathcal{U}^{univ}/\mathcal{A}_{g,d}}\right)^{\otimes r}$ of the Hodge bundle on moduli stack
- Also define compactification $A^*_{g,d}$

Key argument 1 Metric on $\hat{\mathcal{M}}$ has logarithmic singularities along the bdry, Northcott prop in this setting is still true.

Key argument 2 Show that (A,λ) have good reduction outside S

$\Rightarrow h_{\hat{\mathcal{M}}}((A,\lambda)) < C(g,d,S)$

§4 Arakelov intersection theory

V/k variety, smooth

$CH^i(V) := Z^i(V) / \{\operatorname{div}(f) \mid f \in k(Y)^\times,\ Y \subseteq V \text{ of codim } i-1\}$ Chow groups, $0 \le i \le \dim V = n$

$Z^i(V)$: cycles

Intersection pairing: $CH^i(V) \times CH^j(V) \longrightarrow CH^{i+j}(V)$

If V is proper: $\deg: CH^n(V) \longrightarrow \mathbb{Z}$

$$\sum_{\substack{x \in V(k) \\ \text{closed pt}}} n_x \{x\} \longmapsto \sum n_x$$

Well-def'd since for a curve Y, $f \in k(Y)^\times$, $\deg \operatorname{div}(f) = 0$

Arakelov '74, Gillet – Soulé ~'90: define arithmetic Chow groups $\widehat{CH}^i(X)$ for $X \longrightarrow S$ regular projective flat,

$\widehat{CH}^i_{\mathbb{Q}}(X) \times \widehat{CH}^j_{\mathbb{Q}}(X) \longrightarrow \widehat{CH}^{i+j}_{\mathbb{Q}}(X)$ and $\widehat{CH}^{\dim X}_{\mathbb{Q}}(X) \xrightarrow{\widehat{\deg}} \mathbb{R}$

E.g. $\widehat{CH}^1(X) \cong \widehat{Pic}(X)$ and $h_{\hat{\mathcal{L}}}$ is essentially $\widehat{CH}^1_{\mathbb{Q}}(X) \times \widehat{CH}^{\dim X - 1} \to \widehat{CH}^{\dim X}(X) \xrightarrow{\widehat{\deg}} \mathbb{R}$

Rough definition: $\widehat{CH}^i(X) = \left\{ (Z, g) \mid Z \in Z^i(X),\ g \in D^{i-1,i-1}(X) \text{ Green current for } Z \right\} \Big/ \partial A^{i-2,i-1}$, $\bar{\partial} A^{i-1,i-2}$, rational classes

Green current (= dif form with singularities along Z),

$$dd^c g + [\delta_Z] = [\omega_{(Z,g)}] \quad \text{for some smooth } (i,i)\text{-form } \omega_{(Z,g)}$$

Theorems. • Any Z has a Green current. Corollary: $\widehat{CH}^\bullet(X) \twoheadrightarrow CH^\bullet(X)$

• The intersection product is symmetric.

E.g. $\widehat{Pic}(X) \cong \widehat{CH}^1(X)$,

$(\mathcal{L}, \|\cdot\|) \mapsto (\operatorname{div} s,\ \log \|s\|^2)$ for some non-meromorphic $s \in \mathcal{L}_\eta \setminus \{0\}$

The intersection product is roughly given as:

$$(Y, g_Y) \cdot (Z, g_Z) = (Y \cap Z,\ g_Y * g_Z)$$

where Y, Z are assumed to intersect properly,

$g_Y * g_Z = g_Y \wedge \delta_Z + [\omega_Y] \wedge g_Z$ is the star-product of Green currents

1^{st} motivation for lecture:

• Gillet–Soulé theory depends on X, while statements often don't

• arch and nonarch places are treated differently $\Rightarrow$ would be nice to have an analytic theory in the generic fibre, uniformly for all v

§5 Equidistribution results

S.W. Zhang '95, X. Yuan '12

$A/\mathbb{Q}$ ab. var, $\exists$ canonical Néron–Tate height $A(\overline{\mathbb{Q}}) \xrightarrow{h_{NT}} \mathbb{R}_{>0}$

s.t. $h_{NT}(x) \geq 0\ \forall x$ and $h_{NT}(x) = 0$ iff x is a torsion point.

Def. Sequence $\{x_n\}_n \subseteq A(\overline{\mathbb{Q}})$ is small if $h_{NT}(x_n) \longrightarrow 0$.

" is generic if $\forall X \subsetneq A$ subvariety: $\# X \cap \{x_n\}_n < \infty$.

Thm. (Zhang, '95) Let $\{x_n\}_n \subseteq A(\overline{\mathbb{Q}})$ be small, generic. Let $G := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Then $\frac{1}{\# G x_n} \sum_{x \in G x_n} \delta_x \xrightarrow{\text{weakly}} \mu_{can}$ where μ_{can} is the translation invariant probability measure on $A(\mathbb{C})$.

Using this result, one can get the following:

Thm. (S.W. Zhang '95, Ullmo '95, Bogomolov conj.)

$C/\mathbb{Q}$ smooth proper curve, connected, $g(C) \geq 2$.

Then $\exists \varepsilon > 0$ s.t. $\# C(\overline{\mathbb{Q}}) \cap \{x \in \mathcal{J}(\overline{\mathbb{Q}}) \mid h_{NT}(x) < \varepsilon\} < \infty$.

Here $\mathcal{J} := \mathrm{Pic}^0(C)$ is the Jacobian, $C \hookrightarrow \mathcal{J}$ some fixed embedding.

Cor. (Raynaud, Manin-Mumford conj.): $C \cap \mathcal{J}(\overline{\mathbb{Q}})_{\mathrm{tors}}$ is infinite.

2nd motivation. Formulate such results p-adically. Analytic theory enables us to formulate the results of equidistribution theory p-adically.

§6 Analytic Arakelov theory

S.W. Zhang '95: "Adelically metrised line bundle"

Chambert-Loir: "Chambert-Loir measure" '06

Chambert-Loir — Ducros '12: } Theory of (p,q)-forms on Berkovich analytic

Gubler — Künnemann '14: } spaces. + Green currents

+ Intersection products

In the course, we will focus on these.

Rough sketch: $X^{alg}/\mathbb{C}_p$ smooth projective, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$

$X := (X^{alg})^{an}$ Berkovich analytic space $\rightarrow$ path-conn T2 space,

atlas of anal. functions

(not nec. a sheaf)

For $x \in U \subseteq X$; f anal on U get $|f(x)| \in \mathbb{R}_{\geq 0}$

Def. $f: X \longrightarrow \mathbb{R}$ smooth if analytically locally $\exists z_1, \ldots, z_n$ invertible analytic factors and $\exists \varphi: \mathbb{R}^r \longrightarrow \mathbb{R}$ C^∞ s.t. $f = \varphi(-\log|z_1|, \ldots, -\log|z_n|)$

E.g. $f: \mathbb{P}^1(\mathbb{C}_p) \longrightarrow \mathbb{R}$,

$$f(z) = \begin{cases} -\log|p| & \text{if } |z| \leq |p| \\ -\log|z| & \text{if } |p| \leq |z| \leq 1 \\ 0 & \text{if } |z| \geq 1 \end{cases}$$

On $\mathbb{R}^n$ define the doubled de Rham complex: $A^\bullet \otimes_{C^\infty} A^\bullet$,

$$A^p(U) = \sum_{i_1, \ldots, i_p} f_{i_1, \ldots, i_p} \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}, \quad f_I \in C^\infty(U, \mathbb{R})$$

Comes with $d, \partial, \overline{\partial}$.

Define $A^{\cdot,\cdot}$ on X via pullback under smooth maps $X \supseteq U \xrightarrow{\varphi_1,\dots,\varphi_n} \mathbb{R}^n$ and sheafification.

We thus have
- $\int : A^{\dim X, \dim Y}(X) \longrightarrow \mathbb{R}$ with Stokes formula,
- metrised line bundle $(\mathcal{L}, \|\cdot\|)$ on X
  $\Rightarrow$ can def. $c_1(\hat{\mathcal{L}}) \in A^{1,1}(X)$, $c_1(\hat{\mathcal{L}})^n \in A^{n,n}(X)$
- currents, Green currents, Arakelov intersection theory.

II. Adic spaces 1

~~11.10.2018~~
15.10.2018

§1. Motivation

$X/\mathbb{C}$ smooth variety $\longmapsto$ $X(\mathbb{C})$ complex manifold

- Uniqueness of holomorphic continuation: if $U \subseteq X(\mathbb{C})$ open, connected and $f, g \in \mathcal{O}_{X(\mathbb{C})}(U)$ holom., $f|_V = g|_V$ on $V \subseteq U$ for some V open $\Rightarrow f = g$.
- E.g. X projective then $\mathrm{Coh}(X) \xrightarrow{\cong} \mathrm{Coh}(X(\mathbb{C}))$ (GAGA).

We want a similar theory for $X/\mathbb{C}_p$ where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$

Main obstacle: $\mathbb{C}_p = \coprod_{x \in \overline{\mathbb{F}}_p} \mathrm{pr}^{-1}(x)$, $\mathrm{pr} : \mathcal{O}_{\mathbb{C}_p} \longrightarrow \overline{\mathbb{F}}_p$

In particular, if $X/\mathbb{C}_p$ is a variety then $X(\mathbb{C}_p)$ with the p-adic topology is totally disconnected.

§2. Huber rings

Intro: W & S. Berkeley lectures
Wedhorn: Adic Spaces
B. Conrad: ?

More advanced: ?

Def. 1) A topological ring A is adic if $\exists I \subseteq A$ ideal, called an ideal of definition, s.t. $\{I^n \mid n \in \mathbb{N}\}$ is a fundamental system of open nbhds of 0. W(6.1) 5.18

2) A top. ring A is Huber if $\exists A_0 \subseteq A$ open adic s.t. $\exists I \subseteq A_0$ f.g. ideal of defn.

Convention. "A adic" $\Rightarrow$ $\exists$ fin. gen. ideal of defn. Ber 2.2.1

3) A_0 resp. I as in 2) are a ring resp. ideal of defn of A.

Def. A a top. ring, $S \subseteq A$ set. W 5.25, 5.27

1) S is bounded if $\forall\, 0 \in U \subseteq A$ open $\exists\, 0 \in V \subseteq A$ open s.t. $VS \subseteq U$.

Given S, set $S(n) := S \cdots S = \{ s_1 \cdots s_n \mid s_i \in S\}$

2) S is power-bounded if $\bigcup_n S(n)$ is bounded.

$x \in A$ is power-bounded if $\{x\}$ is.

$A^\circ := \{\text{power-bounded elements}\} \subseteq A$

3) S is topologically nilpotent if $S(n) \longrightarrow 0$ in the sense that $\forall\, 0 \in U \subseteq A$ $\exists N$ s.t. $S(n) \subseteq U$ $\forall n \geq N$

$A^{\circ\circ} := \{\text{top. nilp. elts}\} \subseteq A$

Exc. 1) S finite $\Rightarrow$ S bounded.

W 5.28

2) $S_1, \dots, S_n \subseteq A$ bounded $\Rightarrow$ $\bigcup_i S_i$, $S_1 + \dots + S_n$, $S_1 \cdots S_n$ are bounded

3) S_1 pb, S_2 tn $\Rightarrow$ $S_1 S_2$ tn

Exc. A Huber, $I \subseteq A_0$ pair of defn. Then $S \subseteq A$ is bounded $\Leftrightarrow$ $\exists n$: $I^n S \subseteq A_0$.
Moreover, S is top. nilp. $\Leftrightarrow$ $\exists n$: $S(n) \subseteq I$.

Prop. A Huber.

1) $A_0 \subseteq A$ subring is a rg of defn $\Leftrightarrow$ A_0 open & bounded

2) $A^\circ = \bigcup_{\substack{A_0 \subseteq A \\ \text{rg of defn.}}} A_0$

Pf: 1): "$\Rightarrow$" follows from the exercise: openness by def, boundedness by exc.
"$\Leftarrow$" A Huber $\Leftrightarrow$ $\exists\, I, \subseteq B$ pair of defn.

Claim. $C := B[A_0]$ is a ring of defn and IC is a f.g. ideal of defn.

• Boundedness: A_0 bounded $\Rightarrow$ $\exists r$: $I^r A_0 \subseteq B$. $\Rightarrow$ $I^{r+n} C \subseteq I^n$ $\forall n$ (here we used that A_0 is closed under multiplication) ✓

• $IC \subseteq C$ is an ideal of defn; i.e. $(IC)^n$ is a fund. system of nbhds of 0:
$I^n \subseteq (IC)^n \Rightarrow (IC)^n$ open $\forall n$. Also $(IC)^{r+n} \subseteq I^n \Rightarrow \{(IC)^n\}$ cofinal ✓

Claim. A_0 adic with f.g. ideal of defn.

A_0 is open $\Leftrightarrow$ $\exists N$: $J(IC)^N \subseteq A_J$

Let $j_1, \dots, j_n \in J$ be s.t. $J = \sum_{i=1}^n C j_i$, $K := \sum_{i=1}^n A_0 j_i \subseteq A_0$ ideal.

Claim. $J^2 \overset{②}{\subseteq} K \overset{①}{\subseteq} J$. Here ① just follows from the def. of K.

②: Let $j, k \in \mathcal{J}$, write $k = c_1 j_1 + \dots + c_m j_m$, $c_i \in C$

$\Rightarrow jk = (jc_1)j_1 + \dots + (jc_m)j_m$, $\forall jc_i \in \mathcal{J} \subseteq A_0$

This also finishes the proof of 1).

2): $A_0 \subseteq A$ open & bounded, $x \in A^\circ$ $\Rightarrow$ $A_0[x]$ bounded $\overset{1)}{\Rightarrow}$ ideal of defn.

§3 Completions

Def. 1) A Huber ring A is complete if it is Hausdorff and Cauchy sequences converge.

2) A continuous ring homomorphism $A \longrightarrow \hat{A}$ between Huber rings, where $\hat{A}$ is complete, is called completion if it is universal with this property, i.e. $\forall B$ complete Huber, $\forall A \to B$ continuous:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ \hat{A} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Prop. A Huber, $I \subseteq A_0$ pair of defn.

1) There exist $A \longrightarrow \hat{A}$ completion, and $\varprojlim_n (A/I^n, +) = (\hat{A}, +)$.

In particular: $\hat{A}_0 \subseteq \hat{A}$ is an open subring, and $(\hat{A}_0, I\hat{A}_0)$ is a pair of defn of $\hat{A}$.

2)
$$\begin{tikzcd} \hat{A} & \hat{A}_0 \arrow[l] \\ A \arrow[u] & A_0 \arrow[l] \arrow[u] \end{tikzcd}$$
is cocartesian.

Pf: Omitted, see Huber.

§4 Examples of Huber rings

1) (Schemes) A discrete $\Rightarrow$ Huber, in part., $(A, (0))$ is a pair of defn.

2) (Formal schemes) A adic itself. e.g. noetherian adic rings, but also $\mathcal{O}_{\mathbb{C}_p}$ with $(\mathcal{O}_{\mathbb{C}_p}[[z]], (p, z))$

3) (Rigid spaces) Construction: A_0 ring, $g \in A_0$ regular.

Claim. $A := A_0[g^{-1}]$ + topology that makes $\{g^n A_0\}_{n \in \mathbb{N}}$ into a fund. system of open nbhds of 0 is Huber, (A_0, gA_0) is a pair of defn.

Pf: NTS A is a top. ring.

Let $a, b \in g^r A_0$ for some $r \in \mathbb{Z}$. Then $(a + g^n A_0)(b + g^n A_0) = ab + g^n(aA_0 + bA_0) + g^{2n} A_0$

$\subseteq ab + g^{\min(n+r, 2n)} A_0$. □

Def. A Huber ring A is Tate if $\exists g \in A^\times$ top. nilp.

W6.10

Prop. 1) The A constructed above is Tate.

2) Conversely, for a Tate ring A, $g \in A^\times \cap A^{oo}$, $A_0 \subseteq A$ ring of defn $\Rightarrow \exists n$ s.t. $g^n \in A_0$, A_0 ($g^n A_0$) adic and $A = A_0[(g^n)^{-1}]$.

Pf: 1) omitted.

2): A_0 open $\Rightarrow \exists n \geq 0$: $g^n \in I$

Let $I \subseteq A_0$ be an ideal of defn.

g unit, multiplication by g^{-1} is continuous $\Rightarrow g^n A_0 \subseteq A$ is open.

$\Rightarrow \exists r \geq 0$ s.t. $I^r \subseteq g^n A_0 \subseteq I$. $\Rightarrow A_0$ is $g^n A_0$-adic.

We have $A_0[(g^n)^{-1}] \hookrightarrow A$ embedding.

But $\forall x \in A$: $g^m x \longrightarrow 0$ $(m \to \infty)$ $\Rightarrow g^m x \in A_0$ for $m \gg 0$ $\Rightarrow x \in g^{-m} A_0$

Cor. A Huber, and $\mathbb{Q}_p$-algebra, the algebra structure being continuous. Then $\forall A_0 \subseteq A$ ring of defn, the topology on A_0 is p-adic with pA_0 being an ideal of defn.

Exc. $(\mathbb{Q}_p[\![z]\!], +)$ with the topology making $\mathbb{Z}_p[\![z]\!]$ into an open subgp. with (p,z)-adic topology $\longrightarrow$ this is not a top. ring.

Typical Tate ring: K non-archimedean field, $\mathcal{O} \subseteq K$ integers, $0 \neq \pi \in \mathcal{O}$ top. nilp.
Endow $K[z_1, \dots, z_n] = \mathcal{O}[z_1, \dots, z_n][\pi^{-1}]$ with the topology as above.

Def. Tate algebra $K\langle z_1, \dots, z_n\rangle :=$ the completion of $K[z_1, \dots, z_n]$ from above. W 5.54, cf. §5.6

Equals $\left\{ \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n} \;\middle|\; a_{i_1, \dots, i_n} \to 0 \text{ for } \sum_{i=1}^n i_k \to \infty \right\}$

§5 Valuations

For Γ a totally ordered abelian group written multiplicatively, $\Gamma \cup \{0\}$ denotes the ordered monoid s.t. $0 < \gamma$, $\gamma 0 = 0$ $\forall \gamma \in \Gamma$. E.g. $\mathbb{R}_{\geq 0}$

Def. 1) Valuation on a ring A: $|\cdot| : A \longrightarrow \Gamma \cup \{0\}$ for some Γ tot. ordered ab. gp. s.t.
(i) $|ab| = |a||b|$
(ii) $|a+b| \leq \max(|a|, |b|)$
(iii) $|1| = 1$, $|0| = 0$.

2) $|\cdot| : A \to \Gamma \cup \{0\}$, $|\cdot|' : A \to \Gamma' \cup \{0\}$ are equivalent if $\forall a, b \in A$:
$|a| \leq |b| \iff |a|' \leq |b|'$

Equivalently: $\exists$ order-preserving iso $\operatorname{Im}|\cdot| \cong \operatorname{Im}|\cdot|'$ s.t. the following commutes:

$$\begin{tikzcd} & \operatorname{Im}|\cdot| \arrow[dd, "\cong"] \\ A \arrow[ur] \arrow[dr] & \\ & \operatorname{Im}|\cdot|' \end{tikzcd}$$

$\operatorname{Spv}(A) := \{|\cdot| \text{ on } A\} /\cong$

3) For $|\cdot| \in \operatorname{Spv}(A)$, set $\ker|\cdot| := \operatorname{supp}|\cdot| = \{x \in A \mid |x| = 0\} \in \operatorname{Spec} A$.

Ex.
$$\begin{tikzcd} \operatorname{Spv}(A) \arrow[d, "\ker"', shift right] \\ \operatorname{Spec}(A) \arrow[u, bend right, "\text{section}"'] \end{tikzcd} \quad \mathfrak{p} \in \operatorname{Spec}(A) \mapsto |x|_{\mathfrak{p}} = \begin{cases} 1 & x \notin \mathfrak{p} \\ 0 & x \in \mathfrak{p} \end{cases}$$

Def. Valuation ring R: integral domain s.t. $\forall x \in (\operatorname{Frac} R)^\times$, $x \in R$ or $x^{-1} \in R$.

E.g. $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ discrete

$\bigcup_n \mathbb{Z}_p[\zeta_{p^n}]$, $\mathbb{Z}_p[\zeta_{p^\infty}] := \widehat{\bigcup_n \mathbb{Z}_p[\zeta_{p^n}]}_p$, $\mathcal{O}_{\mathbb{C}_p}$ non-discrete

Prop. 1) k a field. Then $\operatorname{Spv}(k) \cong \{R \subseteq k \text{ val. rg.} \mid \operatorname{Frac} R = k\}$

2) In general, $\operatorname{Spv}(A) = \{(\mathfrak{p}, R) \mid \mathfrak{p} \in \operatorname{Spec} A,\ R \subseteq \operatorname{Frac} A/\mathfrak{p} \text{ as above}\}$

Construction for 1): Given $R \subseteq k$, set $\Gamma := k^\times / R^\times$.

Order $aR^\times \leq bR^\times \iff ab^{-1} \in R$

$|\cdot|_R : k \longrightarrow \Gamma \cup \{0\}$, $\quad |a|_R = \begin{cases} aR^\times \in \Gamma & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$

Conversely: $R(|\cdot|) := \{x \in k \mid |x| \leq 1\}$

§6 Continuous valuations

Def. A top. ring. $|\cdot| \in \operatorname{Spv}(A)$ is continuous if $\forall \gamma \in \Gamma_{|\cdot|}$, $\{x \in A \mid |x| < \gamma\} \subseteq A$ is open

$\operatorname{Cont}(A) \subseteq \operatorname{Spv}(A)$ is the set of cont. valuations.

Ex. $\operatorname{Spf} A := \{\mathfrak{p} \subseteq A \text{ open prime ideal}\}$

$$\begin{tikzcd} & \operatorname{Cont} A \arrow[d, "\ker"] \\ \operatorname{Spf} A \arrow[ur, "\mathfrak{p} \mapsto |\cdot|_{\mathfrak{p}}"] \arrow[r] & \operatorname{Spv} A \end{tikzcd}$$

Rmk. Let $\gamma \in \Gamma$. Then by the triangle inequality, $\{x \in A \mid |x| < \gamma\}$ is an abelian subgroup, hence closed. Thus $\ker|\cdot| = \bigcap_{\gamma \in \Gamma} \{x \mid |x| < \gamma\}$ is closed for a cont. valuation $|\cdot|$.

RDFINAG, lecture 2-3

Def. 1) A Huber. A <u>ring of integral elements</u> of A is an $A^+ \subseteq A$ open, integrally closed subring.

2) A <u>Huber pair</u> is (A, A^+) where A is Huber, $A^+ \subseteq A$ is a rg of int. elts.

Rmk. $A^\circ \subseteq A$ integrally closed, hence a ring of integral elts. "maximal"

$\overline{\mathbb{Z}[A^{\circ\circ}]}^{int}$ also int closed "minimal"

3) (A, A^+) Huber pair

$\underline{Spa(A, A^+)} := \{ |\cdot| \in Cont(A) \mid |x| \leq 1 \quad \forall x \in A^+ \}$

III. Adic spaces 2

19.10.2018

Ex. $A = \mathbb{Q}_p[\varepsilon]/\varepsilon^2$ with p-adic topology from $\mathbb{Z}_p[\varepsilon]/\varepsilon^2$

$\Rightarrow A^\circ = \mathbb{Z}_p + \mathbb{Q}_p \cdot \varepsilon$ is not bounded itself.

Other rings of def.: $\mathbb{Z}_p[p^{-n}\varepsilon]$, $n \in \mathbb{N}$, $\Gamma = \mathbb{R}^\times_{>0}$

Def. A <u>map of Huber pairs</u>: $(A, A^+) \xrightarrow{\varphi} (B, B^+)$, φ continuous, $\varphi(A^+) \subseteq B^+$.

Huber rings form a category.

§1 Topology on $Spa(A, A^+)$

$|\cdot|_0 : \mathbb{Q}_p\langle z\rangle \xrightarrow{z \mapsto 0} \mathbb{Q}_p \xrightarrow{|\cdot|_p} \mathbb{R}_{\geq 0}$, supp $|\cdot|_0 = z\mathbb{Q}_p\langle z\rangle$

Recall: $U \subseteq Spec\, A$ is closed iff $U^c = V(I)$ for some $I \subseteq A$ ideal.

Nice opens: $D(f)$, $f \in A$

- quasicompact, • give a bases of the topology
- universal property: $D(f) = \operatorname{Im}(Spec(A \to A_f))$,

  $\forall A \xrightarrow{\varphi} B$: $Spec\,\varphi$ factors over $D(f)$ iff $\varphi(f) \in B^\times$

$\Rightarrow$ can define canonically $\mathcal{O}_{Spec A}(D(f)) = A_f$.

Def. (A, A^+) Huber. The <u>topology on $Spa(A, A^+)$</u> is generated by $U(\frac{T}{s})$ for $T \subseteq A$ finite, $s \in A$, where $\underline{U(\frac{T}{s})} := \{ x \in Spa(A, A^+) \mid |t(x)| \leq |s(x)| \neq 0 \quad \forall t \in T \}$

Exc. $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ map of Huber pairs $\Rightarrow$ Spec φ is continuous.

Def. X top space is <u>spectral</u> if

1) X quasicompact (qc) & quasiseparated (qs)
2) X sober
3) X has a base of qc opens.

Thm/Rmk. X is spectral iff $\exists R$ ring, $Spec\, R \cong X$ homeomorphism.

↙as everything else

Thm. (Huber) $\mathrm{Spa}(A, A^+)$ is spectral.

Rmk. A Huber, $\hat{A}$ completion. Then $A^+ \longrightarrow \widehat{A^+}$ (here $\widehat{A^+} = \varprojlim_{n \gg 0} A^+/I$ for $I \subseteq A_0$ pair of defn.) is a bijection $\{A^+ \subseteq A \text{ ring of int. elts}\} \cong \{\widehat{A^+} \subseteq \hat{A} \text{ ring of int elts}\}$

Also, $\widehat{A^+} = \overline{\mathrm{Im}(A^+ \to \hat{A})}$, where $\overline{}$ is the topological closure.

Prop. $\forall (A^+, A)$ the map $(A, A^+) \longrightarrow (\hat{A}, \widehat{A^+})$ induces a homeomorphism on adic spectra:

$$\mathrm{Spa}(\hat{A}, \widehat{A^+}) \xrightarrow{\cong} \mathrm{Spa}(A, A^+).$$

Prop. (adic Nullstellensatz) Let (A, A^+) be a complete Huber pair.

1) $A \neq 0 \iff \mathrm{Spa}(A, A^+) \neq \emptyset$

2) $A^+ = \{f \in A \mid |f(x)| \leq 1 \quad \forall x \in \mathrm{Spa}(A, A^+)\}$

3) $f \in A$ is invertible $\iff |f(x)| \neq 0 \quad \forall x \in \mathrm{Spa}(A, A^+)$.

§2 Rational opens

Construction. (A, A^+) Huber, $T = \{t_1, \dots, t_n\} \subseteq A$ s.t. TA is an open ideal of A, $s \in A$.

$A\left(\frac{T}{s}\right) := A\left[\frac{1}{s}\right]$ with topology s.t. $A_0\left[\frac{T}{s}\right]$ is open, adic with ideal of defn. $IA_0\left[\frac{T}{s}\right]$

(This is indep. of the choice $T \subseteq A_0 \subseteq A$.)

$A\left(\frac{T}{s}\right)^+ := \overline{A^+\left[\frac{T}{s}\right]}^{\,\mathrm{int}}$

Thm. (see e.g. Berkeley notes) $\left(A\left(\frac{T}{s}\right), A\left(\frac{T}{s}\right)^+\right)$ is Huber,

- $A \longrightarrow A\left(\frac{T}{s}\right)$ is initial among $A \xrightarrow{\varphi} B$ maps of Huber rings, s.t. $\varphi(s) \in B^\times$ and $\varphi\left(\frac{t}{s}\right) \in B^\circ \quad \forall t \in T$.
- $(A, A^+) \longrightarrow (A(T/s), A(T/s)^+)$ is initial among $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ maps of Huber pairs, $\varphi(s) \in B^\times$, $\varphi(t/s) \in B^\circ \quad \forall t \in T$.

Rmk/Def. $(A\langle T/s\rangle, A\langle T/s\rangle^+) :=$ completion of $(A(T/s), A(T/s)^+)$

There is a similar univ. prop. with B complete.

Def. $U \subseteq \mathrm{Spa}(A, A^+)$ is a rational domain if $\exists T \subseteq A$ finite s.t. $TA \subseteq A$ open, $\exists s \in A$ s.t. $U = U(T/s)$.

Thm. • $T \subseteq A$ finite, $TA \subseteq A$ open, $s \in A$. Then

$$\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+) \longrightarrow X = \mathrm{Spa}(A, A^+) \text{ is a homeo onto } U(T/s).$$

• In particular, rational domains are qc.

- $U' \subseteq U(T/s)$ is rational in X $\Leftrightarrow$ rational for $(A\langle T/s\rangle, A\langle T/s\rangle^+)$
- Given $\varphi: (A, A^+) \longrightarrow (B, B^+)$. Then Spec φ factors via $U(T/s)$ $\Leftrightarrow$

  $\Leftrightarrow$ φ factors (nec. uniquely) over $(A\langle T/s\rangle, A\langle T/s\rangle^+)$. univ. prop.

Def. $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) := (A\langle T/s\rangle, A\langle T/s\rangle^+)$. only depends on U up to canonical iso.

$\forall V \subseteq U$ rationals we get $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \xrightarrow{\text{res}^U_V} (\mathcal{O}_X(V), \mathcal{O}_X^+(V))$

§3 Example of $\mathbb{D}$

$K/\mathbb{Q}_p$ alg closed, complete non-archimedean, e.g. $K = \mathbb{C}_p$,

$\mathcal{O} := K^\circ$ ring of integers

Fix $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$ nontrivial, giving K its topology.

Then $\mathrm{Spa}(K, \mathcal{O}) = \{|\cdot|\}$ is a 'geometric point'

Def. $\mathbb{D} := \mathrm{Spa}(K\langle z\rangle, \mathcal{O}\langle z\rangle) (= \mathrm{Spa}(K[z], \mathcal{O}[z]))$ closed unit disc over $\mathrm{Spa}(K, \mathcal{O})$

$\mathbb{D}: \{\text{Huber pairs} / (K, \mathcal{O})\}^{op} \longrightarrow$ Set functor, $\mathbb{D}(A, A^+)$.

$$\mathrm{Hom}_{(K,\mathcal{O})}((K\langle z\rangle, \mathcal{O}\langle z\rangle), (A, A^+)) \cong A^+$$
$$\varphi \mapsto \varphi(z)$$

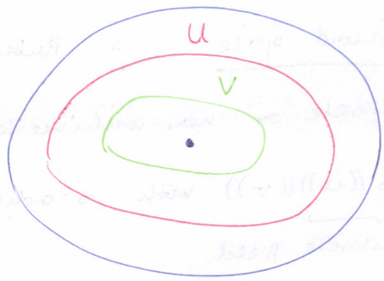
Ex. $\mathbb{D}(K, \mathcal{O}) = \mathcal{O} = \{a \in K \mid |a| \leq 1\}$

$\mathcal{O} \hookrightarrow \mathbb{D}$

$$a \longmapsto \left[\begin{matrix} K\langle z\rangle & \longrightarrow & K & \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0} \\ z & \longmapsto & a & \end{matrix}\right]$$

Fact. The subspace topology on $\mathcal{O}$ is the p-adic one.

Typical rational domains



$$V = U\left(\frac{\{z, p^n\}}{p^n}\right) = \{x \mid |z(x)| \leq |p^n(x)|\},$$

here $\neq 0$ and $|p^n(x)| \leq |p^n(x)|$ are automatic

e.g. $V \cap \mathcal{O} = \{a \in \mathcal{O} \mid |a| \leq |p^n|\}$

$$U = U\left(\frac{p^n}{z}\right) = \{x \mid |p^n(x)| \leq |z(x)| \neq 0\}$$

$U \cap \mathcal{O} = \{a \in \mathcal{O} \mid a \notin p^n \cdot \mathfrak{m}_{\mathcal{O}}\}$ If $n = 0 \Rightarrow U \cap \mathcal{O} = \mathcal{O}^\times$.

Exc. $\left(\mathcal{O}_{\mathbb{D}}(V), \mathcal{O}^+_{\mathbb{D}}(V)\right) = \left\{ \sum_{i\geq 0} a_i z^i \;\middle|\; p^{-ni} a_i \to 0 \text{ as } i \to \infty \right\} \cong \left(K\langle z\rangle, \mathcal{O}\langle z\rangle\right)$

$\left(\mathcal{O}_{\mathbb{D}}(U), \mathcal{O}^+_{\mathbb{D}}(U)\right) = \left\{ \sum_{i\in\mathbb{Z}} a_i z^i \;\middle|\; a_i \to 0 \text{ when } i \to \infty,\ p^{+ni} a_i \to 0 \text{ when } i \to -\infty \right\}$

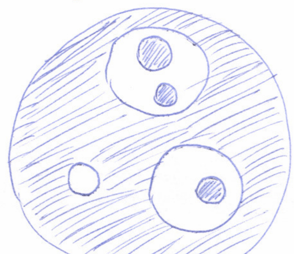
Thm (Weierstrass preparation thm., ref. BGR)

Let $f \in K\langle z\rangle$. Then $\exists g \in K[z],\ u \in \mathcal{O}\langle z\rangle^\times$ s.t. $f = gu$

Get: $\{|f| \leq |p^n|\} = \coprod_{\text{fin}}$ "closed" discs

$\{|p^n| \leq |f|\} = \mathbb{D} \setminus \left(\coprod_{\text{fin}} \text{closed discs}\right)$

In general: a rational U expresses as $U \cong \coprod_{\text{fin}} \left(\mathbb{D} \setminus \coprod_{\text{fin}} \text{closed discs}\right)$



U is the union of the shraffed sets

Addendum to §2: Thm. Rational opens $\subseteq \operatorname{Spa}(A, A^+)$ form a basis of topology.

§4 Affinoid fields

Def. k field, $|\cdot| : k \to \Gamma \cup \{0\}$. The valuation topology on k induced from $|\cdot|$ is the top. generated by $\left\{ |\cdot|^{-1}(\gamma < \delta) \;\middle|\; \delta \in \Gamma \right\}$

Def. The top. field k is non-archimedean if a non-trivial $|\cdot| : k \to \mathbb{R}_{\geq 0}$ defines the top.

Def. A valuation $|\cdot| : k \to \Gamma \cup \{0\}$ (resp. its valuation subring $\Gamma_{|\cdot|}$) is microbial if its valuation topology is non-archimedean.

Def. An affinoid field is a Huber pair (k, k^+) where k is a topological field, either discrete or non-archimedean, and $k^+ \subseteq k$ is its valuation subring.

Ex. $k = \underbrace{\mathbb{F}_p((u))}_{\text{viewed as a discrete field}}((v))$ with v-adic topology on $\mathbb{F}_p((u))[[v]]$

$$\begin{aligned} \operatorname{ord}_v : k &\longrightarrow \mathbb{Z} \cup \{\infty\} \\ f &\longmapsto \operatorname{ord}_v(f) \end{aligned}$$

This k is a non-archimedean field

$R := \{x \in k \mid \operatorname{ord}_v(x) \geq 0\} = \mathbb{F}_p((u))[[v]]$

$\cup\!\!\!\mid$

$\mathfrak{m} = v\,\mathbb{F}_p((u))[[v]]$

$R/m = \mathbb{F}_p((u)) \supseteq r := \mathbb{F}_p[[u]]$ $\qquad R' := pr^{-1}(r) \subseteq R$

$$\begin{array}{c} R' \\ \cup\!\text{l} \\ \mathbb{F}_p[[u]] + v\,\mathbb{F}_p((u))[[v]] \end{array}$$

Exc. $R' \subseteq k$ is a valuation ring

$\Rightarrow$ we get a valuation $k \longrightarrow \underbrace{k^\times/(R')^\times \cup \{\infty\}}$

$@ \quad \cong$

$\mathbb{Z}\times\mathbb{Z}$ with lexicographic ordering

$@: \; f \longmapsto \left(\mathrm{ord}_v(f),\ \mathrm{ord}_u\left(\underbrace{v^{-\mathrm{ord}_v(f)}}_{\text{this cancels the } v\text{-power}} f\right)\right)$ $\qquad (n,m) > (n',m')$ if $n > n'$ or $n = n'$ & $m > m'$

Claim. v' induces the same topology on k as ord_v. In particular, v' is microbial.

Pf: $v \in k$ is a topologically nilpotent unit, i.e. $v \in k^\times \cap k^{\circ\circ}$,

and both ord_v and v' give k the topology with fundamental set of
opens of 0 given by $\{g^n R\}$

In $\mathbb{Z}\times\mathbb{Z}$, $(1,0)$ is cofinal, i.e. $\forall \gamma \in \mathbb{Z}\times\mathbb{Z}\ \exists n$ s.t. $n(1,0) > \gamma$.

Top. from v' on k is gen'd by $\left\{\{x \in k \mid v'(x) \geq (n,m)\}_{(n,m)}\right\} =$

$= \left\{\{v'(x) \geq (n,0)\}_n\right\} = \left\{v^n R' \mid n \in \mathbb{N}\right\}$

But $vR \subseteq R' \subseteq R \Rightarrow \{v^n R'\}_n$ and $\{v^n R\}_n$ gen. the same top. on k $\square$

IV Adic spaces 3

22.10.2018

Exc./Rmk. k non-archimedean $\Rightarrow$ k Huber, even Tate.

If $k^+ \subseteq k$ s.t. (k,k^+) affinoid $\Rightarrow$ k^+ is microbial and induces the given top. on k.

Conversely: k top field, Huber, valuation v induces the topology $\Rightarrow$ k is discrete or non-archimedean and v is microbial.

Prop. R valuation ring. TFAE:

1) R microbial
2) R has a prime ideal of height 1
3) $\exists \eta \in R\setminus\{0\}$ topologically nilpotent wrt. the valuation topology.

§1 Specialisations in $\mathrm{Spa}(k,k^+)$

Def. X spectral, $x,y \in X$, x specialises to y if $y \in \overline{\{x\}}$

Notation: $x \rightsquigarrow y$, $x \leq y$

Suppose (k, k^+) to be an affinoid field.

Prop. $x, y \in \mathrm{Spa}(k, k^+)$, $R_x, R_y \subseteq k$ valuation rings. Then $x \rightsquigarrow y \iff R_y \subseteq R_x$.

Pf: Consider $U\left(\frac{f}{g}\right) = \{ |f(z)| \leq |g(z)| \neq 0 \}$ for some $f, g \in k$

Then $y \in U\left(\frac{f}{g}\right) \iff fg^{-1} \in R_y \Rightarrow fg^{-1} \in R_x$ $\square$

Exc. $k^+ \in R_x, R_y, R_z \subseteq k$ valuation rings, $R_x \subseteq R_y, R_z$. Then $R_y \subseteq R_z$ or $R_z \subseteq R_y$.
In other words, given $y \in \mathrm{Spa}(k, k^+)$: $\{x \mid x \rightsquigarrow y\}$ form a chain under the generalisation relation.

Prop. $k^+ \subseteq R_y \subseteq k$ val. ring of $y \in \mathrm{Spa}(k, k^+)$. Then $\mathrm{Spec}\, R_y \cong \{ R_y \subseteq R' \subseteq k \text{ valuation ring} \}$
$$\mathfrak{f} \mapsto R_{y,\mathfrak{f}}$$

Assume k to be non-archimedean. Then y has a unique rank 1 generalisation
$\iff \exists$ height 1 prime ideal as in the Prop, $\mathfrak{f} = k^{oo}$. Then $R_{y,\mathfrak{f}} = k^o$.

Construction (up to equivalence) of this valuation:

Choose $\gamma \in k^\times \cap k^{oo}$, $|\cdot|_\gamma : k \longrightarrow \mathbb{R}_{\geq 0}$, $|f|_\gamma := 2^{-\inf\left\{\frac{1}{m} \,\middle|\, \gamma^{-m} f^m \in k^o\right\}}$

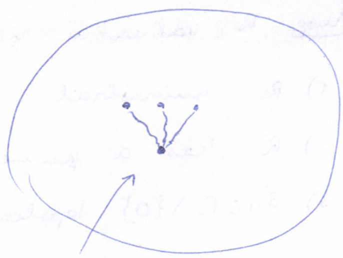
Def. $v : A \longrightarrow \Gamma \cup \{\infty\}$ is of rank $\leq d$ if $\exists\, \Gamma \hookrightarrow (\mathbb{R}_{>0})^{\times d}$ order-preserving embedding where the codomain is endowed with the lexicographic order.
v is of rank d if of rank $\leq d$ but not of rank $\leq d+1$.

Prop. (k, k^+) affinoid field, $x \in \mathrm{Spa}(k, k^+)$, $R_x \subseteq k^o$, $m_x \subseteq R_x$ max. ideal.
Then
$$\{y \mid x \rightsquigarrow y\} \simeq \{k^+ \subseteq R_y \subseteq R_x \subseteq k\}$$
$$\simeq \{ k^+/m_x \subseteq r_x \subseteq R_x/m_x \text{ val. subring} \}$$
$$\simeq \mathrm{Spa}\big(\underbrace{R_x/m_x}_{\text{discrete topology}},\ k^+/m_x\big)$$

Ex. We have seen $\mathrm{Spa}\left(\mathbb{F}_p((u))((v)),\ \mathbb{F}_p[\![u]\!] + v\, \mathbb{F}_p((u))[\![v]\!] \right) = \{ \vdots \}$

$\mathrm{ord}_v \longleftrightarrow \mathbb{F}_p((u))[\![v]\!] \longrightarrow \mathbb{F}_p((u)) \supseteq r = \mathbb{F}_p[\![u]\!]$

Ex. $\mathrm{Spa}(\mathbb{F}_p(u)((v)),\ \mathbb{F}_p[u] + v\mathbb{F}_p(u)[\![v]\!]) \simeq |\mathbb{A}^1_{\mathbb{F}_p}| = \mathrm{Spa}(\mathbb{F}_p(u), \mathbb{F}_p[u])$



this looks like a microbe

§2 Residue fields / specialisations

(A, A^+) pair, $X \subseteq \mathrm{Spa}(A, A^+)$

Def. Local ring at $x \in X$ is $\mathcal{O}_{X,x} := \varinjlim_{\substack{x \in U \\ \text{rational}}} \mathcal{O}_X(U)$. Here $\varinjlim$ is taken in Ring, not in Top.

Induced valuation $|\cdot(x)| : \mathcal{O}_{X,x} \longrightarrow \Gamma_x \cup \{0\}$, this equips $\mathcal{O}_{X,x}$ with a valuation topology.

$\mathcal{O}_{X,x} \supseteq \mathfrak{m}_x = \operatorname{supp} |\cdot(x)| = \{ f \in \mathcal{O}_{X,x} \mid |f(x)| = 0 \}$ maximal ideal

Prop. $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is a local ring.

Def. $\kappa(x) := \mathcal{O}_{X,x} / \mathfrak{m}_x$ with valuation and topology from $|\cdot(x)|$.

$\kappa(x)^+ :=$ the induced valuation ring

Then $(\kappa(x), \kappa(x)^+)$ is an affinoid field. (@)

Def. $x \in X$ is analytic if $\kappa(x)$ is non-archimedean.

$X_a := \{ x \in X \text{ analytic} \}$

Prop. • @, i.e. $(\kappa(x), \kappa(x)^+)$ is an affinoid field.

• x is analytic $\Leftrightarrow$ $\exists x \in U$ rational open s.t. $\mathcal{O}_X(U)$ is Tate.

$\Leftrightarrow$ $\kappa(x)^+$ is microbial.

Cor. $X_a \subseteq X$ is open.

Def. A specialisation $x \rightsquigarrow y$ is vertical if $\operatorname{supp} x = \operatorname{supp} y$. (i.e. they lie in the same fibre above the spectrum.)

Prop. Let $x \in X_a$. Then $(A, A^+) \longrightarrow (\kappa(x), \kappa(x)^+)$ induces a homeomorphism $\mathrm{Spa}(\kappa(x), \kappa(x)^+) \hookrightarrow X_a$ onto $\overline{\{x\}} \cap X_a$.

In particular, any specialisation in X_a is vertical.

Note: There are also horizontal specialisations. It can be shown that any specialisation can be obtained by successive horizontal and vertical specialisations.

§3 Points of $\mathbb{D}$

K non-archimedean complete field.

$\mathcal{O} := K^\circ \quad (\Rightarrow \operatorname{Spa}(K, \mathcal{O}) = \{pt\})$

$k := \mathcal{O}/\mathfrak{m}$

Choose $|\cdot| : K^\times \longrightarrow \mathbb{R}_{\geq 0}$ valuation giving K its topology,

$\forall r \in (0,1] \cap |K^\times|$ some $\pi_r \in \mathcal{O}$ s.t. $|\pi_r| = r$

$\mathbb{D} = \operatorname{Spa}(K\langle z\rangle, \mathcal{O}\langle z\rangle) = \operatorname{Spa}(K[z], \mathcal{O}[z])$ Details: Conrad, lecture 11.

Thm. Points of $\mathbb{D}$ come in 5 types, described as follows:

(1) classical points: $x \in \mathbb{D}$ s.t. $\kappa(x)/K$ is finite

(2) Gauss-like points: $\left(\kappa(x)^+/\mathfrak{m}_x\right)/k$ is transcendental and $\sqrt{v_x(K)} = \sqrt{v_x(\kappa(x))}$
where v_x is the valuation corresponding to x.

(3) $\left(\kappa(x)^+/\mathfrak{m}_x\right)/k$ is algebraic and $\sqrt{v_x(K)} \subsetneq \sqrt{v_x(\kappa(x))}$

(4) $\left(\kappa(x)^+/\mathfrak{m}_x\right)/k$ is algebraic and $\sqrt{v_x(K)} = \sqrt{v_x(\kappa(x))}$

works w/o divisible hull in the alg. closed case

All points of type (1–4) are of rk 1.

(5) points of rk 2.

Moreover,

- (3) exists $\Leftrightarrow \sqrt{|K^\times|} \subsetneq \mathbb{R}_{\geq 0}$
- (4) exist $\Leftrightarrow \hat{K}$ is not spherically closed
- (1), (3–5) are closed. The only specialisations are (2) $\rightsquigarrow$ (5).

From now on, $K = \overline{K}^{alg}$.

Type (1): let $x \in K$, $v_x : K[z] \longrightarrow K \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$, $z \longmapsto x$ is a valuation.

Lemma. v_x is continuous $\Leftrightarrow x \in \mathcal{O}$.

Pf: $x \in \mathcal{O} \Rightarrow v_x(\pi^n \mathcal{O}[z]) \subseteq v_x(\pi^n \mathcal{O}) = \{ r \in |K| \mid r \leq |\pi^n| \} \Rightarrow$ continuous

$x \in K \setminus \mathcal{O} \Rightarrow v_x(x^{-n} z^n) = 1 \ \forall n$ but $x^{-n} z^n \to 0 \ (n \to \infty)$. □

$\Rightarrow \{\text{Type } (1)\} \cong \mathcal{O} \hookrightarrow \mathbb{D}$, $x \longmapsto v_x$

Type (2): $r \in |K^\times| \cup \{0\}$, $x \in K$.

$v_{x,r} : K[z] \longrightarrow \mathbb{R}_{\geq 0}$

$\sum_n a_n z^n \longmapsto \max_n \{ |a_n| r^n \}$ is a valuation

Lemma. $v_{x,r}$ continuous $\iff$ $r\leq 1$, $x\in \mathcal{O}$

Pf: If $r\leq 1$, $x\in\mathcal{O}$ then $v_{x,r}(\pi^n \mathcal{O}[z]) \subseteq \{ s\in |K| \mid s\leq |\pi^n|\}$ $\Rightarrow$ continuous

If $r>1$, $x\in\mathcal{O}$ choose $a\in K^\times$ s.t. $1<|a|<r$. Then $a^{-1}(z-x)$ is top. nilpotent, but $v_{x,r}(a^{-1}(z-x))>1$.

If $x\in K\setminus \mathcal{O}$, use $v_{x,0}\subseteq v_{x,r}$ and the argument from before. □

Prop. All type (2) pts are of the form $v_{x,r}$, $x\in\mathcal{O}$, $r\in |\mathcal{O}|\setminus\{0\}$.

Rmk.
- $v_{x,1}$ is the Gauss point η
- $v_{x,0}=v_x$
- $v_{x,r}=v_{x',r'}$ $\iff$ $r=r'$ and $|x-x'|\leq r$

Def. $\mathbb{D}(x,r):=\mathcal{U}\left(\frac{z-x}{\pi_r}\right)=\{ y \mid |(z-x)(y)|\leq |\pi_r(y)|\}$ disc of radius r, center x in $\mathbb{D}$

Then $\mathbb{D}(x,r)=\mathbb{D}(x',r')$ $\iff$ $r=r'$ and $|x-x'|\leq r$.

Prop. (BGR, Prop. 5.1.4.3., Maximum modulus principle)

$\forall f\in K\langle z\rangle$: $|f(v_{x,r})|=\sup\limits_{y\in \mathbb{D}(x,r)\cap\mathcal{O}}|f(y)|$, and this sup is actually a max.

For x type (2): $\kappa(x)^+/m_x\cong k(z)$.

The img of $\mathcal{O}\langle z\rangle^+=\begin{cases} k & \text{if } r<1\\ k[z] & \text{if } x=\eta\end{cases}$ $\Rightarrow$ $\overline{\{v_{x,r}\}}\cong\begin{cases}\mathbb{P}^1_k & r<1\\ \mathbb{A}^1_k & x=\eta\end{cases}$

This yields (5) too.

Type (3): $r\in(0,1]\setminus|K|$, $x\in\mathcal{O}$, $v_{x,r}(f):=\sup\limits_{y\in\mathcal{O},\,|y-x|\leq r}|f(y)|$ (not a max)

Type (4): $x_n\in\mathcal{O}$, $r_n\in(0,1]$, $r_n\downarrow r'>0$, $\mathbb{D}(x_n,r_n)\supsetneq\mathbb{D}(x_{n+1},r_{n+1})$,

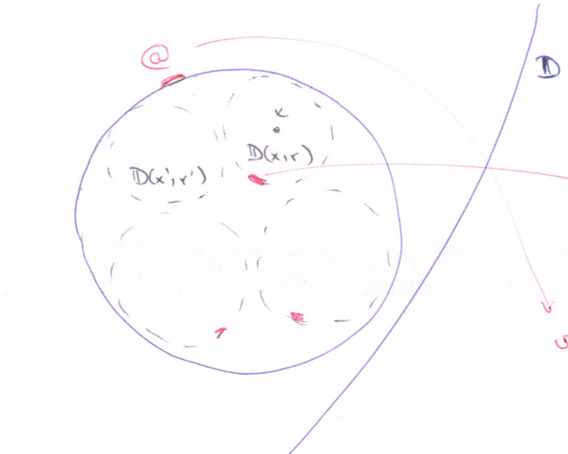
$\mathcal{O}\cap\bigcap\limits_n \mathbb{D}(x_n,r_n)=\emptyset$. Define $v_{(x_n)(r_n)}:=\inf\limits_n v_{x_n,r_n}$, this is a type (4) valuation.

Def. $r\in|K^\times|$, $r\leq 1$, $x\in\mathcal{O}$. $\mathring{\mathbb{D}}(x,r):=\bigcup\limits_{\substack{r'<r\\ r'\in|K^\times|}}\mathbb{D}(x,r')$ open disc inside $\mathbb{D}$.

$\mathring{\mathbb{D}}(x,r)$ is not qc, in particular not affinoid/rational.

$$\mathbb{D}(x,r)=\underbrace{\coprod_{x'\in(x+\pi_r\mathcal{O})/\pi_r m}}_{k=\mathcal{O}/m \text{ many}}\mathring{\mathbb{D}}(x',r)\ \amalg\left(\overline{\{v_{x,r}\}}\setminus\{v^\infty_{x,r}\}\right)$$

$v^\infty_{x,r}$ corresponds to $\infty\in\mathbb{P}^1_k$



characterised by $v(x_r^{-1}(z-x)) < 1$

but $\pi_r^{-1}(z-x)$ not top nilpotent

$v_@$ is char'd by $v_@\left(\pi_r^{-1}(z-x)\right) < 1$

but $\pi_r^{-1}(z-x)$ is still power-bounded

E.g. $\underbrace{\{x \mid |z(x)|=1\}}_{U(\frac{z}{1})} \,\dot\cup\, \mathring{\mathbb{D}}(0,1)$ is not a covering of $\mathbb{D}$,

even though it covers $\mathcal{O} \subseteq \mathbb{D}$

The pt x s.t. $|z(x)|<1$ but $z \in \mathcal{K}(x)$ is not top. nilp is not covered.

Rmk. 29.10. Faltings: Shimura Vtyes, MPI (may be a waste of time)

26.10.2018

V Adic spaces 4

§1 Adic spaces

Def. $\mathcal{V}$:= cat. of $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ where

- $(X, \mathcal{O}_X)$ is a locally topologically ringed space, i.e. $\mathcal{O}_X$ a sheaf of complete rings, $\forall\, \mathcal{U} = \bigcup_i \mathcal{U}_i$: $\mathcal{O}_X(\mathcal{U}) \hookrightarrow \prod_i \mathcal{O}_X(\mathcal{U}_i)$ closed, $\mathcal{O}_{X,x}$ local ring $\forall x \in X$
- $v_x : \mathcal{O}_{X,x} \longrightarrow \Gamma_x \cup \{0\}$ val. up to equivalence (no continuity condition), $\mathcal{O}_{X,x} = \varinjlim^{\mathrm{Ring}}_{x\in\mathcal{U}} \mathcal{O}_X(\mathcal{U})$

Morphisms $(X, \mathcal{O}_X, \{v_x\}_x) \longrightarrow (Y, \mathcal{O}_Y, \{v_y\}_y)$ are $(f, f^b) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ map of loc. top. ringed spaces, $f^b : \mathcal{O}_Y \to f_* \mathcal{O}_X$, such that

- $\forall x \in X$: $v_{f(x)}$ and $\mathcal{O}_{Y,f(x)} \xrightarrow{f^b} \mathcal{O}_{X,x} \xrightarrow{v_x} \Gamma_x \cup \{0\}$ are equivalent.

Def. (A, A^+) Huber pair, $X = \mathrm{Spa}(A, A^+)$, $W \subseteq X$ open, $\mathcal{O}_X(W) := \varprojlim_{\mathcal{U}\subseteq W \text{ rational}} \mathcal{O}_X(\mathcal{U})$

We call (A, A^+) sheafy if the presheaf $\mathcal{O}_X$ is a sheaf of top. rings.

For (A, A^+) sheafy we immediately get $\mathrm{Spa}(A, A^+) \in \mathcal{V}$

Def. Affinoid adic spaces: the essential image. (all the objects that are iso to an object in $\mathrm{Spa}(A, A^+)$)

Def. Adic space: $(X, \mathcal{O}_X, \{v_x\}) \in \mathcal{V}$ that has a finite afinoid covering.

Thm. (Huber) (A, A^+) Huber pair is sheafy (at least) in 3 cases:

1) (Schemes) A discrete
2) (Formal schemes) A f.g. over noetherian ring of defn, (e.g. noeth + adic)
3) (Rigid spaces) A Tate and strongly noetherian, i.e. $A\langle z_1, \dots, z_n\rangle = \left\{ \sum_I a_I x^I \;\middle|\; a_I \to 0 \text{ as } |I| \to \infty \right\}$ are noetherian $\forall n$

RDFINAG, lecture 5

Def. X adic space. The <u>sheaf of integral functions</u> is

$$\mathcal{O}_X^+(U) := \{ f \in \mathcal{O}_X(U) \mid v_x(f) \leq 1 \quad \forall x \in U \}$$

By adic Nullstsatz, for sheafy (A, A^+): $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(\mathrm{Spa}(A,A^+)) = A^+$.

Recall: $\begin{array}{rcl} \mathrm{Rng}^{op} & \longrightarrow & \mathrm{Sch} \\ A & \longmapsto & \mathrm{Spec}(A) \end{array}$ is fully faithful and $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec} A) = \mathrm{Hom}_{\mathrm{Rng}}(A, \mathcal{O}_X(X))$

We have a similar result for adic spaces.

Thm. $\{\text{complete sheafy Huber pairs}\} \longrightarrow \mathrm{Ad}$ (cat. of adic spaces)

$$A \longmapsto \mathrm{Spa}(A, A^+)$$

is a fully faithful functor and has the global sections functor as an adjoint:

$$\mathrm{Hom}_{\mathcal{V}}(X, \mathrm{Spa}(A, A^+)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Huber}}\big((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X))\big)$$

Def/Rmk. X adic space, $x \in X$. We get

- <u>residue field</u> $(\kappa(x), \kappa(x)^+)$
- notion of "x <u>analytic</u>": $X_a = \{x \in X \text{ analytic}\} \subseteq X$ open. See p. 17.

§2 Adic and finite type morphisms

"I hope this section does not discourage you from working with adic spaces."

Ref.: Huber Ch. 12

Def. A, B Huber, $f : A \longrightarrow B$ is <u>adic</u> if $\exists I \subseteq A_0 \subseteq A$ is a pair of definition, $\exists B_0 \subseteq B$ s.t. $f(A_0) \subseteq B_0$ and $f(I) B_0$ is an ideal of definition.

Def. $f : X \to Y$ morphism of adic spaces is <u>adic</u> if $\forall x \in X$ $\exists x \in U \subseteq X$, $V \subseteq Y$ both open affinoid s.t. $f(U) \subseteq V$, $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \longrightarrow (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is adic. @

Rmk. $f : A \to B$ & A Tate $\Rightarrow$ B Tate.

Recall: $x \in X$ is analytic if $x \in U \subseteq X$ open affinoid nbh. s.t. $\mathcal{O}_X(U)$ is Tate.

Prop. TFAE, for $f : X \to Y$:

(1) f is adic

(2) $f(X_a) \subseteq Y_a$

(3) @ but for all U, V affinoid for which $f(U) \subseteq V$

Def. $f : A \to B$ map of Huber ~~spaces~~ rings is <u>topologically of finite type</u> if f is adic and $\exists M \subseteq B$ finite set s.t. $f(A)[M]$ is dense in B, and $\exists A_0, B_0$, $f(A_0) \subseteq B_0$, $N \subseteq B_0$ finite, $f(A_0)[N]$ is dense in B_0.

Def. $f: X \to Y$ is <u>locally weakly of finite type</u> if $\forall x \in X$ $\exists x \in U \subseteq X$ $\exists f(U) \subseteq V \subseteq Y$ s.t. $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is topologically of finite type.

Rmk. For A Tate, $f: A \to B$ is top oft $\iff$ $\exists n$ and a factorisation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ A\langle z_1, \dots, z_n\rangle \arrow[ur, "g"'] & \end{tikzcd}$$

with g open and surjective (and, of course, continuous). This looks like oft in classical algeo.

Def. $f: (A, A^+) \to (B, B^+)$ map of complete Huber pairs is <u>topologically of finite type</u> $f: A \to B$ is top oft and $\exists C \subseteq B^+$ open s.t. B^+ is integral over C, $f(A^+) \subseteq C$, $f: A^+ \to C$ is top oft.

Def. $f: X \to Y$ is <u>locally of finite type</u> if $\forall x \in X$ $\exists x \in U \subseteq X$, $\exists f(U) \subseteq V \subseteq Y$ s.t. $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is top. oft.

Def. $f: (A, A^+) \to (B, B^+)$ is a <u>quotient map</u> if $f: A \to B$ surjective, open and B^+ is integral over $f(A^+)$.

Rmk. A Tate, $f: (A, A^+) \to (B, B^+)$ is top. oft $\iff$ $\exists n$ and a factorisation

$$\begin{tikzcd} (A, A^+) \arrow[r, "f"] \arrow[d] & (B, B^+) \\ (A\langle z_1, \dots, z_n\rangle, A^+\langle z_1, \dots, z_n\rangle) \arrow[ur, "g"'] & \end{tikzcd}$$

where g is a quotient map.

<u>Warning</u>. Ad does <u>not</u> admit arbitrary fiber products, as opposed to Sch.

Prop. Let $\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$ be given. Then $X \times_Z Y$ exists if (1) f oft OR
(2) f loft & g adic.

§ 3 Rigid analytic spaces

K non-arch field, $\mathcal{O} = K^\circ \subseteq K$

Classical **Def.** (BGR, Ch. 6.) A <u>K-affinoid algebra</u> is a K-algebra of the form $K\langle z_1, \dots, z_n\rangle / I$.

Prop.
- (Hilbert Basissatz) $K\langle z_1, \dots, z_n\rangle$ noetherian $\forall n$
- $\forall I \subseteq K\langle z_1, \dots, z_n\rangle$ ideal is closed. In particular, $K\langle z_1, \dots, z_n\rangle / I$ is a complete Huber ring.

Rmk. $\{K\text{-affinoid algebras}\} = \{\text{Huber-}K\text{-algebras top oft}\}$

Cor. Any (A, A^+) with A top oft $/K$ is sheafy.

Historical Rmk. For A K-affinoid, consider $\mathrm{Sp}\, A = \mathrm{MSpec}\, A = $ type (1) pts of $\mathrm{Spa}(A, A^+)$
Define a Grothendieck topology gen'd by rational subsets on $\mathrm{Sp}\, A$.

Endow with a sheaf of rings as we did for adic spaces. Note that we don't get a sheaf on Sp A, we only get to evaluate on rational subsets.

We obtain, by def., an affinoid rigid analytic space /K.

Glue along admissible opens, thus obtain rigid analytic spaces /K.

Thm. The functor $(\mathrm{Sp}\, A, \dots) \longrightarrow \mathrm{Spa}(A, A^\circ)$ yields a functor

$$\{\text{rigid analytic spaces } /K\} \longrightarrow \{\text{adic spaces } X \text{ loft } /\mathrm{Spa}(K, \mathcal{O})\}$$
$$\cup\!\!|$$
$$\{\text{qs rigid analytic spaces } /K\} \xrightarrow{\simeq} \{\text{qs loft } X\ /\mathrm{Spa}(K, \mathcal{O})\}$$
$$X \longmapsto X^{ad} \quad (\text{adification})$$

Exc. $\mathbb{D} = \mathrm{Spa}(K\langle z\rangle, \mathcal{O}\langle z\rangle)$. Then $\mathbb{D} \amalg \mathbb{D} / \{z_1 \neq 0\} \overset{id}{\cong} \underbrace{\{z_2 \neq 0\}}$ is qc but not qs adic space.

not admissible open
$\Rightarrow$ does not come from a rigid space.

Also, $\mathbb{D} \amalg \mathbb{D} / \{|z_1| = 1\} \overset{id}{\cong} \{|z_2| = 1\}$ is qcqs.

Rmk. $\{\text{Sheaves on } X\} \xrightarrow{\simeq} \{\text{Sheaves on } |X^{ad}|\}$

with Grothendieck topology

top. space

§4 Some examples

$\pi \in \mathcal{O}^{\circ\circ} \cap K^\times$, $\mathcal{O} \subseteq K$ as before

Test object (A, A^+) sheafy complete $/(K, \mathcal{O})$

- $\mathbb{D} = \mathbb{D}(0,1) = \mathrm{Spa}(K\langle z\rangle, \mathcal{O}\langle z\rangle)$

$$\mathbb{D}(A, A^+) = \mathrm{Hom}_{\mathrm{Huber}}\big((K\langle z\rangle, \mathcal{O}\langle z\rangle), (A, A^+)\big) = A^+$$
$$f^b \mapsto f^b(z)$$

- $\mathring{\mathbb{D}}(0,1) = \bigcup_{0<r<1} \mathbb{D}(0,r) \subseteq \mathbb{D}$ open adic subspace

$$\mathring{\mathbb{D}}(A, A^+) = \bigcup_n \{a \in A^\times \mid \pi^{-1} a^n \in A^+\} = A^{\circ\circ}$$

Def. X adic space is partially proper if $X(A, A^+) = X(A, A^\circ)$ $\forall (A, A^+)$ sheafy complete.

Ex. $\mathbb{D}$ is not partially proper, $\mathring{\mathbb{D}}$ is.

- $\mathbb{A}^{1,ad} = \bigcup_n \mathbb{D}(0, |\pi|^{-n})$ adic affine line

Here $\mathbb{D}(0, |\pi|^{-n}) = \mathrm{Spa}(K\langle z\rangle, \mathcal{O}\langle z\rangle)$, transition map $\mathbb{D}(0, |\pi|^{-n}) \longrightarrow \mathbb{D}(0, |\pi|^{-n+1})$

$$\pi z \longleftarrow z$$

$\mathbb{A}^{1,ad}(A,A^+) = \bigcup_n \pi^{-n} A^+ = A$ In particular, this is partially proper.

Rmk. Consider $\mathbb{D}(0,1) \underset{\varphi_1}{\hookrightarrow} \mathbb{D}(0,1) \underset{\varphi_2}{\hookrightarrow} \mathbb{D}(0,1) \underset{\varphi_3}{\hookrightarrow} \dots$ ascending chain of discs

There are 4 possibilities.

1) "radius goes to ∞": $\bigcup_\varphi \mathbb{D}(0,1) \cong \mathbb{A}^1$

2) "radius becomes constant": $\bigcup \mathbb{D}(0,1) \cong \mathbb{D}$

3) "strictly monotone & converges against r":

3a) $r \in |K^\times| \Rightarrow \bigcup \mathbb{D}(0,1) \cong \mathring{\mathbb{D}}$.

3b) $r \in \mathbb{R}_{>0} \setminus \sqrt{|K^\times|} \Rightarrow \bigcup \mathbb{D}(0,1) \cong \mathbb{D}(0,r)$

- Annulus: $A(1,1) = \mathrm{Spa}(K\langle z,z'\rangle, \mathcal{O}\langle z,z'\rangle) \cong U\left(\frac{1}{z}\right) \subseteq \mathbb{D}$

Exc. Define $A(r_1,r_2)$ annulus of inner radius r_1, outer radius r_2, $r_1 < r_2$;
$\mathring{A}(r_1,r_2)$

- Multiplicative group: $\mathbb{C}_m^{ad} = \mathbb{A}^{1,ad} \setminus \underbrace{\{z=0\}}_{\text{exc.: this is a point}}$

$\mathbb{C}_m^{ad}(A,A^+) = A^\times$

- Fibre products of all these (they exist in the current setting)

VI. Adic spaces 5

29.10.2015

Recall: $X/\mathbb{C}$ scheme locally oft $\longmapsto X^{an}/\mathbb{C}$ analytic space, cx. mfold w/ singularities and a morphism $(X^{an}, \mathcal{O}_{X^{an}}) \longrightarrow (X, \mathcal{O}_X)$ which is universal in the sense that $\forall Y/\mathbb{C}$ analytic $\forall (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ factors over $(X^{an},\mathcal{O}_{X^{an}})$:

$$\begin{tikzcd} (Y,\mathcal{O}_Y) \arrow[rr] \arrow[dr, dashed, "\exists !"'] & & (X,\mathcal{O}_X) \\ & (X^{an},\mathcal{O}_{X^{an}}) \arrow[ur] & \end{tikzcd}$$

This $(X^{an},\mathcal{O}_{X^{an}})$ is called the analytification.

If $X/\mathbb{C}$ smooth then $X^{an} = X(\mathbb{C})$ is a complex manifold.

§1 Analytification

K non-archimedean, $\mathcal{O} = K^\circ$

Convention. • If A is Huber then a pseudouniformiser $\pi \in A$ is an element $\pi \in A^{\circ\circ} \cap A^\times$.
(If A has a pseudouniformiser then A is Tate.)

- $\mathrm{Spa}(A) := \mathrm{Spa}(A, A^\circ)$

Let $\pi \in \mathcal{O}$ be a pseudouniformiser

Thm. $X/\mathrm{Spec}\,K$ scheme loft. Then there is an adic space $(X^{ad}, \mathcal{O}_{X^{ad}}, \{v_x\})/_{\mathrm{Spa}\,K}$ and a morphism $(X^{ad},\mathcal{O}_{X^{ad}}) \to (X,\mathcal{O}_X)$ of loc. ringed spaces such that for

$$\begin{tikzcd} (X^{ad},\mathcal{O}_{X^{ad}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & (X,\mathcal{O}_X) \arrow[d] \\ \mathrm{Spa}\,K \arrow[r] & \mathrm{Spec}\,K \end{tikzcd}$$

every adic space Y and morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ we have a factorisation

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[rr] \arrow[dr, dashed, "\exists!"'] & & (X, \mathcal{O}_X) \\ & (X^{ad}, \mathcal{O}_{X^{ad}}) \arrow[ur] & \end{tikzcd}$$

X^{ad} is called the adification of X.

Examples. • $\left(\mathbb{A}^n_{\operatorname{Spec} K}\right)^{ad} = \mathbb{A}^{n, ad}_{\operatorname{Spec}(K, \mathcal{O})}$

• $\underbrace{\operatorname{Spec}\left(K[z_1, \dots, z_n]/(f_1, \dots, f_r)\right)}_{X}{}^{ad}$. By def this represents a sheafy complete

$(A, A^+)/(K, \mathcal{O}) \longmapsto \{(a_1, \dots, a_n) \in A^n \mid f_i(\underline{a}) = 0 \ \forall i\}$

$\Rightarrow$ given as $X^{ad} \subseteq \mathbb{A}^{n, ad}_K$ s.t. $X^{ad} \cap \mathbb{D}^n(0, |\pi|^{-s}) \cong \operatorname{Spa} K\langle z_1, \dots, z_n\rangle / \left(f_i(\pi^{-s} z_1, \dots, \pi^{-s} z_n)\right)$

i.e. $X^{ad} = \bigcup_s \operatorname{Spa} K\langle z_1, \dots, z_n\rangle / \left(f_i(\pi^{+s} z_1, \dots, \pi^{+s} z_n)\right)$ with transition map $\pi z_i \mapsto z_i$.

• General X^{ad} by gluing.

E.g. $\mathbb{P}^{n, ad}_K$ and for $V_+(I) \subseteq \mathbb{P}^n_K$ get $V_+(I)^{ad} \subseteq \mathbb{P}^{n, ad}_K$

Def. (A, A^+) Huber pair, $I \subseteq A$ ideal. The quotient is $\left(A/I, \overline{A^+/A^+ \cap I}^{\,int}\right)$.

Globalisation: if X is an adic space, $\mathcal{I} \subseteq \mathcal{O}_X$ qcoh sh. of ideals then a closed adic subspace $Z = V(\mathcal{I}) \subseteq X$ is defined by

i) $Z := \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$

ii) $\mathcal{O}_Z := \mathcal{O}_X / \mathcal{I}$

iii) v_z, for $z \in Z$, is def'd via $\mathcal{O}_{Z,z} = \mathcal{O}_{X,x}/\mathcal{I}_x \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x \xrightarrow{v_x} \Gamma_x \setminus \{0\}$

Prop. $(Z, \mathcal{O}_Z, \{v_z\})$ def'd above is an adic space; if locally $X \cong \operatorname{Spa}(A, A^+)$ & $\mathcal{I} \in \Gamma(X, \mathcal{I})$ then it is given by $\operatorname{Spa}\left(A/\mathcal{I}, \overline{A^+/A^+ \cap \mathcal{I}}^{\,int}\right)$.

Def. $f: Z \to X$ is a closed immersion if iso onto some $V(I) \subseteq X$.

Prop. If $Y \hookrightarrow X/\operatorname{Spec} K$ closed immersion, $X/\operatorname{Spec} K$ loft def'd by $\mathcal{I} \subseteq \mathcal{O}_X$, then $Y^{ad} \hookrightarrow X^{ad}$ closed immersion def'd by $\mathcal{I}\mathcal{O}_{X^{ad}}$.

§2 Example $\mathbb{P}^{n,ad}_K$

Def. Vector bundle on an adic space X: loc free $\mathcal{O}_X$-module loc of fin. rank.

Thm. (Kedlaya – Liu) (A,A^+) sheafy complete Huber pair.

Then $\{\text{proj } A\text{-modules}\} \cong \{\text{vector bundles on } \mathrm{Spa}(A,A^+)\}$

$$M \longmapsto \mathcal{O}_{\mathrm{Spa}(A,A^+)} \otimes_K M$$
$$\Gamma(\mathrm{Spa}(A,A^+),\mathcal{E}) \longleftarrow \mathcal{E}$$

Prop. 1) $\mathbb{P}^{n,ad}_K \in \mathrm{Ad}$ represents a functor $X/_{\mathrm{Spa}\,K} \longrightarrow \{\mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}\}/\cong$

where $\mathcal{L}$ is a line bundle, i.e. vb of rk 1.

2) Let $U_i \cong \mathbb{A}^{n,ad}_K \subseteq \mathbb{P}^{n,ad}_K$ standard opens wrt choice of coordinates in $\mathbb{P}^n$.

Then $\mathbb{P}^{n,ad}_K = \bigcup_{i=0}^{n} \varphi_i(D_i)$.

Pf: 1) is formal since $(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K})$ represents given functor in LRS

And the def of vb only depends on the lrs underlying the adic space.

2) Consider $x \in |\mathbb{P}^{n,ad}_K|$, image of the special point of $\mathrm{Spa}(\kappa(x),\kappa(x)^+) \xrightarrow{x} \mathbb{P}^{n,ad}_K$

(a.o.u.). $x \longmapsto \kappa(x)^{n+1} \longrightarrow \kappa(x) \longrightarrow [x_0,\dots,x_n] \in \mathbb{P}^n(\kappa(x))$

Since $\kappa(x)^\times/(\kappa(x)^+)^\times$ is totally ordered, we get i s.t.

Then $[x_0,\dots,x_n] \sim [x_i^{-1}x_0,\dots,x_i^{-1}x_n] \in \varphi_i(D_i)$. Essentially, we used the valuative criterion for properness. $\square$

§3 Relation to noetherian formal schemes

Recall. A noetherian & adic $\Rightarrow$ (A,A) sheafy

Prop. $\mathfrak{X}$ locally noetherian formal scheme $\Rightarrow$ $\exists\, \mathfrak{X}^{ad}$ adic and $(\mathfrak{X}^{ad},\mathcal{O}^+_{\mathfrak{X}^{ad}}) \xrightarrow{sp} (\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$

map of top loc ringed spaces s.t. $\forall Z$ adic space

$$\begin{tikzcd} (Z,\mathcal{O}_Z) \arrow[r] \arrow[d, dashed, "\exists!"] & (\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \\ (\mathfrak{X}^{ad},\mathcal{O}^+_{\mathfrak{X}^{ad}}) \arrow[ur] & \end{tikzcd}$$

The notation sp stands for "specialisation map".

This induces a fully faithful functor $\{\text{loc noeth formal sch}\} \longrightarrow \mathrm{Ad}$.

Let $(K,\mathcal{O},\pi)$ be as before.

Def. $\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}$ is locally formal oft if locally of the form $\mathrm{Spf}\,A$ where $A/\mathcal{O}$ oft.

Assume $\mathcal{O}$ to be noetherian, i.e. $\mathcal{O}$ is a DVR.

Then any $\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}$ formal scheme, locally formal oft is loc noetherian

$\Rightarrow \mathfrak{X}^{ad}/_{\mathrm{Spa}(\mathcal{O},\mathcal{O})}$ is locally oft

Construction. $\mathfrak{X} = \bigcup_i \mathrm{Spf}\, A_i \Rightarrow \mathfrak{X}^{ad} = \bigcup_i \mathrm{Spa}(A_i, A_i)$

By the construction and the prev lecture fibre prod exist:

$$\begin{tikzcd}
\mathfrak{X}^{ad}_\eta := \mathfrak{X}^{ad} \times_{\mathrm{Spa}\,\mathcal{O}} \mathrm{Spa}\,K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X}^{ad} \arrow[d] \\
\mathrm{Spa}(K,\mathcal{O}) \arrow[r] & \mathrm{Spa}(\mathcal{O},\mathcal{O})
\end{tikzcd}$$

Here η denotes the generic fibre.

Def. $\mathfrak{X}^{ad}_\eta$ is the adic generic fibre of $\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}$.

Examples. • $\widehat{\mathbb{A}^1_\mathcal{O}}$ where $\widehat{\ }$ is the π-adic completion, i.e. this is $\mathrm{Spf}\,\mathcal{O}\langle z\rangle$

For $(R,R^+)/(K,\mathcal{O})$ sheafy complete we get

$$\mathrm{Mor}\left(\mathrm{Spa}(R,R^+),\ \mathrm{Spa}\,\mathcal{O}\langle z\rangle\right) = \mathrm{Hom}_{\text{Continuous } Rg}\left(\mathcal{O}\langle z\rangle, R^+\right) = R^+$$

$\Rightarrow \left(\widehat{\mathbb{A}^1_\mathcal{O}}\right)^{ad}_\eta \cong \mathbb{D}(0,1)_{(K,\mathcal{O})}$. sp: $|\mathbb{D}(0,1)_K| \longrightarrow |\mathbb{A}^1_{\mathcal{O}/m}|$

$x \longmapsto x \bmod m$

sp is continuous, $\mathrm{sp}^{-1}(\underbrace{(z-x)}_{\text{closed pt in } \mathbb{A}^1}) = \mathring{\mathbb{D}}(x,1) \underset{\text{Set}}{=} \mathring{\mathbb{D}}(0,1) \cup \{\text{one type (5) point}\}$

• $\mathfrak{X} = \mathrm{Spf}\,\mathcal{O}[[z]]$, $\mathcal{O}[[z]]$ endowed with the (π,z)-adic topology

Then $\mathfrak{X}^{ad}_\eta(R,R^+) = R^{oo} \Rightarrow \mathfrak{X}^{ad}_\eta \cong \mathring{\mathbb{D}}(0,1)_K$, which is in particular not affinoid!

So this does not behave like it would for schemes.

More
• General construction: $\mathfrak{X} = \mathrm{Spf}\,A/\mathrm{Spf}\,\mathcal{O}$, $A = \left(\mathcal{O}[z_1,\dots,z_n]\right)^{\wedge}_{I}$ for $I \subseteq \mathcal{O}[z_1,\dots,z_n]$, $I = (f_1,\dots,f_r,\pi)$

1) $\mathfrak{X}^{ad}_\eta(R,R^+) = \mathrm{Hom}_{\mathrm{Top}\,Rg}(A,R^+) = \{x_1,\dots,x_n \in (R^+)^n,\ \forall f_i(x_j) \text{ top. nilpotent}\}$

$\Rightarrow \mathfrak{X}^{ad}_\eta = \bigcup_i U\left(\frac{f_1^i,\dots,f_r^i}{\pi}\right) \subseteq \mathbb{D}^n$

§4 Extension to non-noetherian formal schemes

$K, \mathcal{O}, \pi$ as usual, $\mathcal{O}$ not necess. discrete

Problem. It is not known if $(\mathcal{O}[[z]], \mathcal{O}[[z]])$ is sheafy. $\Rightarrow$ the above won't work

Rmk. • SW: Moduli of p-div grps, def of pre-adic spaces. This we won't do.

Def. $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ formal scheme of type (S) if locally $\cong \operatorname{Spf} A$ where A is

(1) noetherian OR

(2) $\exists s \in A$ s.t. $sA \subseteq A$ is an ideal of defn. & $A\left(\frac{A}{s}\right)$ is strongly noetherian

Ex. $\mathfrak{X}/\operatorname{Spf}\mathcal{O}$ is admissible if π-adic, locally formal oft, $\underbrace{\text{flat }/\mathcal{O}}$

this won't matter in the generic fibre

Prop. $(X, \mathcal{O}_X)$ of type (S). Then $\exists$ analytic adic space $(\mathfrak{X}^{ad}_a, \mathcal{O}_{\mathfrak{X}^{ad}_a}, \{v_x\})$ and $(\mathfrak{X}^{ad}, \mathcal{O}_{\mathfrak{X}^{ad}}) \longrightarrow (\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ map of loc top ringed spaces s.t.

for analytic

a) $sp^{-1}(I)\,\mathcal{O}_{\mathfrak{X}^{ad}_a} = \mathcal{O}_{\mathfrak{X}^{ad}_a}$ $\forall I \subseteq \mathcal{O}_\mathfrak{X}$ defining $\mathfrak{X}_{red}$

b) $\operatorname{Im}(\mathcal{O}_\mathfrak{X} \longrightarrow sp_* \mathcal{O}_{\mathfrak{X}^{ad}_a}) \subseteq sp_* \mathcal{O}^+_{\mathfrak{X}^{ad}_a}$ s.t. $(\mathfrak{X}^{ad}_a, \mathcal{O}^+_{\mathfrak{X}^{ad}_a}) \to (\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is a map of locally ringed spaces.

c) Universal as before, with a), b).

Rmk. If $\mathfrak{X} = \operatorname{Spf} A$, A of type (S) (2) then

$$\mathfrak{X}^{ad}_a = \operatorname{Spa}\left(A\left\langle \tfrac{s}{s}\right\rangle, A\left\langle \tfrac{s}{s}\right\rangle^+\right) = \operatorname{Spa}\left(A\left[\tfrac{1}{s}\right]^\wedge, \bar{A}^{int}\right)$$

Example. $X/\operatorname{Spec}\mathcal{O}$ loft, $\mathfrak{X}$ its π-adic completion is of type (S)

$\Rightarrow \mathfrak{X}^{ad}_\eta := \mathfrak{X}^{ad}_a \longrightarrow (\operatorname{Spf}\mathcal{O})^{ad}_a = \operatorname{Spa}(K, \mathcal{O})$

Get natural map $\mathfrak{X}^{ad}_\eta \longrightarrow (X \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} K)^{ad}$. E.g. for $X = \mathbb{A}^1_\mathcal{O}$, get $\mathbb{D}(0,1) \hookrightarrow \mathbb{A}^{1,ad}$ $= \mathbb{P}^{1,ad} \setminus \underbrace{\overline{\mathbb{D}}(\infty, 1)}_{sp^{-1}(\{\infty\})}$

Prop. $X \longrightarrow \operatorname{Spec}\mathcal{O}$ proper, $\mathfrak{X}$ π-adic completion. Then $\mathfrak{X}^{ad}_\eta \xrightarrow{\sim} (X \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} K)^{ad}$ is iso.
Moreover if $U \subseteq X$ is Zariski-open, $\mathcal{U}$ π-adic completion then

$$\mathcal{U}^{ad}_\eta = sp^{-1}\left(U \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} \mathcal{O}/m\right)$$

VII Tropicalisation

Fix a non-archimedean complete valued field, $|\cdot|: K \to \mathbb{R}_{\geq 0}$

§1 Smooth functions

$X/\mathrm{Spa}(K)$ loft, $\quad z: |X| \longrightarrow \mathbb{R}$ continuous

Then $\forall r \in \mathbb{R}$: $z^{-1}(r) \subseteq |X|$ is closed. Hence $(x \rightsquigarrow y) \Rightarrow \big(z(x) = z(y)\big)$

Def. • $X_{max} := \{x \in X \mid v_x \text{ is of rk } 1\}$ maximal relative to specialisations

• $X \longrightarrow X_{max}$

$x \longmapsto \tilde{x} :=$ rk 1 generalisation

Endow X_{max} with the quotient topology. Note that $X_{max} \overset{incl}{\hookrightarrow} X$ is in general not continuous.

Lemma. (Huber 8.1) i) X qc $\Rightarrow X_{max}$ qc

ii) X taut (i.e. qs & $\forall U \subseteq X$ qc open: $\bar{U}$ is qc) $\Rightarrow X_{max}$ T2 and hence $X \longrightarrow X_{max}$ is the maximal T2 quotient

iii) $X = \mathrm{Spa}(A, A^+) \Rightarrow X_{max}$'s topology is generated by $\Big\{\{\tilde{x} \mid f(\tilde{x}) < g(\tilde{x})\} \,\Big|\, f, g \in A\Big\}$

Rmk. if $X = (X^{Berk})^{ad}$ for X^{Berk} T2 strict k-analytical space then X taut & $|X^{Berk}| = X_{max}$

Ex. $\mathbb{D}(0,1) \underset{incl}{\overset{p}{\rightleftarrows}} \mathbb{D}(0,1)_{max}$

$U := \{\tilde{x} \mid |f(\tilde{x})| < 1\} \subseteq \mathbb{D}(0,1)_{max} \quad \Rightarrow \quad p^{-1}(U) = \mathring{\mathbb{D}}(0,1)$

$\pi \in K$ pseudouniformiser, $\mathrm{incl}^{-1}\big(\mathbb{D}(0,|\pi|)\big)$ is closed, its complement: $\{\tilde{x} \mid |f(\tilde{x})| > |\pi|\}$

$p^{-1}\Big(\mathrm{incl}^{-1}\big(\mathbb{D}(0,|\pi|)\big)\Big) = \overline{\mathbb{D}(0,|\pi|)}$ not open

Def/Convention. • For $\tilde{x} \in X_{max}$ $\exists!$ embedding $\mathrm{Im}(v_{\tilde{x}}) \hookrightarrow \mathbb{R}_{\geq 0}$ s.t.

$$\begin{tikzcd} K \arrow[d] \arrow[dr, "|\cdot|"] & \\ \mathcal{O}_{X,\tilde{x}} \arrow[r, "v_{\tilde{x}}"'] & \mathbb{R}_{\geq 0} \end{tikzcd}$$

From now on let $|\cdot(\tilde{x})|$ be normalised like this.

• For $f \in \mathcal{O}_X(X)$ let $|f|: X \longrightarrow \mathbb{R}$

$x \longmapsto |f(\tilde{x})| =: |f|(x)$

Prop. $\forall f \in \mathcal{O}_X(X)$: $|f|$ is continuous

Pf: f defines $X \longrightarrow \mathbb{A}^{1,ad}$, enough to consider $|t|^{-1}\big((a,b)\big) = \begin{cases} \mathring{\mathbb{D}}(0,b) & \text{if } a < 0 \\ \mathring{A}(a,b) & \text{if } 0 \leq a \end{cases}$

where t is the coordinate on $\mathbb{A}^1$.

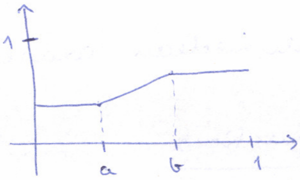
Def. $z: |X| \longrightarrow \mathbb{R}$ is smooth if locally on X of the form $\varphi \circ (|f_1|, \ldots, |f_n|)$ for some $f_i \in \mathcal{O}_X^\times$ and $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth.

Let C^∞ be the sheaf of smooth functions.

Ex. Fix $0<a<b<1$, $a,b\in\sqrt{|K^\times|}$.

$$\mathbb{D}(0,1)=\mathbb{D}(0,a)\cup\mathbb{D}(a,b)\cup\mathbb{D}(b,1)=:U_1\cup U_2\cup U_3$$

Then $z: x\mapsto\begin{cases}|a| & x\in U_1\\ |x| & x\in U_2\\ |b| & x\in U_3\end{cases}$ is smooth,



Well-defined: $U_1\cap U_2=A(a,a)$, $U_2\cap U_3=A(b,b)$.

Exc. $X/\mathrm{Spa}\,K$ loft, $z\in C^\infty(X)$, $x\in X(\hat{\bar{K}})$ $\Rightarrow$ $\exists$ open nbhd $x\in U\subseteq X$: $z|_U=z(X)$

§2 Outlook/Motivation

Want a formalism of (p,q)-forms

Idea. Given a chart $(U,z_1,\dots,z_n)$ with $U\subseteq X$ open, $z=(z_1,\dots,z_n):U\to\mathbb{R}^n$ smooth, $\omega\in A^{p,q}(\mathbb{R}^n)$, $z(\omega)$ should define an element of $A^{p,q}(U)$, e.g. $dz_{i_1}\wedge\dots\wedge dz_{i_p}\wedge d\bar{z}_{i_1}\wedge\dots\wedge d\bar{z}_{i_p}$

Difficulty: compare $z^*\omega$, $\tilde{z}^*\omega$ for different charts $(U,z_1,\dots,z_n)$ and $(U,\tilde{z}_1,\dots,\tilde{z}_n)$.

Idea. Endow $(-\log|f_1|,\dots,-\log|f_n|)(U)\subseteq\mathbb{R}^n$ for $f_1,\dots,f_n\in\mathcal{O}_X(U)^\times$ with piecewise linear structure that "supports" diff. forms.

§3 Tropicalisation

Def. $X/\mathrm{Spa}\,K$ loft. Moment map: $f=(f_1,\dots,f_n):X\longrightarrow\mathbb{G}_m^{n,\mathrm{ad}}$

Tropicalisation map of a moment map wrt. f:

$$\mathrm{trop}_f: X\longrightarrow\mathbb{R}^n$$
$$x\longmapsto(-\log|f_1|,\dots,-\log|f_n|)(x)$$

Note. There is a factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{trop}_f"'] & & \mathbb{G}_m^{n,\mathrm{ad}} \arrow[dl, "\mathrm{trop}_{\mathrm{id}}"] \\ & \mathbb{R}^n & \end{tikzcd}$$

Def. • Polyhedron $P\subseteq\mathbb{R}^n$: $\exists\, l_1,\dots,l_r\in\mathbb{R}^{n,\vee}$ (dual), $c_1,\dots,c_r\in\mathbb{R}$: $P=\bigcap_{i=1}^r\{l_i\geq c_i\}$

• $\dim P:=\dim_{\mathbb{R}}$ (the affine space spanned by P).

• Polyhedral set $M\subseteq\mathbb{R}^n$: $\exists\, P_1,\dots,P_s$ polyhedra, $M=\bigcup_{i=1}^s P_i$

• $\dim M:=\max\{\dim P \mid P\subseteq M \text{ polyhedron}\}$

• $\Gamma:=\log\sqrt{|K^\times|}\subseteq\mathbb{R}$. Then the polyhedron P is Γ-rational if $\exists\, l_i\in\mathbb{Q}^{n,\vee}$, i.e. $l_i=\sum_{i=1}^n a_i x_i^\vee$ with $a_i\in\mathbb{Q}$, and $\exists\, c_i\in\Gamma$ s.t. $P=\bigcap_i\{l_i\geq c_i\}$

• A polyhedral set M is Γ-rational if $M=\bigcup_i P_i$, P_i Γ-rational polyhedra.

Thm. (Bieri-Groves 84, Ducros 03, Berkovich 04, ? 06)

1) $X^{\mathrm{alg}}\subseteq\mathbb{G}_m^n$ closed of pure dim d. Then $\mathrm{trop}(X^{\mathrm{alg},\mathrm{ad}})\subseteq\mathbb{R}^n$ is a Γ-rat polyhedral set of pure dim d, i.e. $\forall x\in M$: $\dim_x M=d$.

2) $X/\mathrm{Spa}\,K$ oft (in part, qc), $f:X\longrightarrow\mathbb{G}_m^{n,\mathrm{ad}}$ moment. Then $\mathrm{trop}_f(X)$ is qc Γ-rat polyh set of $\dim\leq\dim X$

§4 Example of $\mathbb{C}_m^n$

Claim. $\mathbb{C}_m^{n,ad} \longrightarrow \mathbb{R}^n$ surjective

Indeed, for $(r_1,\dots,r_n) = r \in \mathbb{R}^n$, the Gauss point p_r of $\mathbb{D}(0,r_1)\times\dots\times\mathbb{D}(0,r_n)\cap\mathbb{C}_m^{n,ad}$ has img r.

p_r is given on $K[t_1^{\pm 1},\dots,t_n^{\pm 1}]$ by $p_r\left(\sum_i a_i t_1^{i_1}\dots t_n^{i_n}\right) = \max_i\left(|a_i| \exp(-i_1 r_1)\dots\exp(-i_n r_n)\right)$

Ch-D summarise facts about this in §2.2

Why is it Γ-rational? Γ-rational compact polyhedra correspond to generalised annuli $\subseteq \mathbb{C}_m^{n,ad}$

Def. P as above, $U(P) := \mathrm{int}\left(\mathrm{trop}^{-1}(p)\right)$ generalised annulus

Claim. This is affinoid: if $P = \bigcap_{j=1}^{r}\{\ell_j \geq c_j\}$, $\ell_j = \sum_{i=1}^{n_j} a_{ij} x_i^{\nu}$, $a_{ij}\in\mathbb{Z}$, $r_i r_j = -\log|\pi_j|$,

$\pi_j \in K^\times$ then $U(P) = \{x \mid \forall i: |\pi t^{a_i}(x)|^{n_j} \leqq |\pi_j|\}$

Pf: Choose $\varepsilon > 0$ s.t. $P \subseteq [-\log\varepsilon, \log\varepsilon]^n$. Then $U(P)\subseteq A(\varepsilon,\varepsilon^{-1})^{\times n}$ and V is affinoid rational in $A(\varepsilon,\varepsilon^{-1})^{\times n}$.

Moreover $U(P)(\hat{K}) = V(\hat{K})$.

Thm (Huber: Cont. valuations, Thm. 4.1–4.3) $U, V \subseteq \mathrm{Spa}(A, A^\circ)$ rational, A/K top oft, $U(\hat{K}) = V(\hat{K})$. Then $U = V$.

If $W \subseteq U(P)$ open rational in $A(\varepsilon,\varepsilon^{-1})^{\times n}$ then $V\cap W$ is rational and $(W\cap U)(\hat{K}) = W(\hat{K})$

By Thm.: $V \cap W = W$, i.e. $W \subseteq V$.

Cor. For any $M \subseteq \mathbb{R}^n$ Γ-rat polyh set $\exists U \subseteq \mathbb{C}_m^{n,ad}$ open s.t. $\mathrm{trop}(U) = M$.

§5 Hypersurfaces

$X^{alg} := V(f) \subseteq \mathbb{C}_m^n$, $f := \sum_{i\in\mathbb{Z}^n} a_i t_i^{i_1}\dots t_n^{i_n}$, $X := X^{alg,ad}$

Thm (Einsedler–Kapranov–Lind, Thm. 2.2.5)

1) If $X^{alg}\subseteq \mathbb{C}_m^n$ subvariety then $\mathrm{trop}|X| = \overline{\mathrm{trop}\,|X^{alg}(\hat{K})|}^{\text{top. closure}}$

2) Similarly if $X/\mathrm{Spa}\,K$ oft.

Note. $\mathrm{trop}\left(X(\hat{K})\right) \subseteq \Gamma^n$, $\Gamma \subseteq \mathbb{R}$ is dense

Def. $\tau_f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$x \longmapsto \min\{\underbrace{-\log(x_i) + i_1 x_1 + \dots + i_n x_n}_{\text{linear } \mathbb{R}^n \to \mathbb{R}} \mid i\}$

Thm. $\mathrm{trop}(X(\hat{K})) = \{x \in \Gamma^n \mid \text{minimum in } \tau_f(x) \text{ taken by} \geq 2 \text{ of lin. functionals}\}$

$\mathrm{trop}(X) = \{x \in \mathbb{R}^n \mid \tau_f \text{ not diff in } x\}$

Pf: Let $a_1, \dots, a_n \in V(f)(\hat{K})$, $x_i := -\log|a_i|$

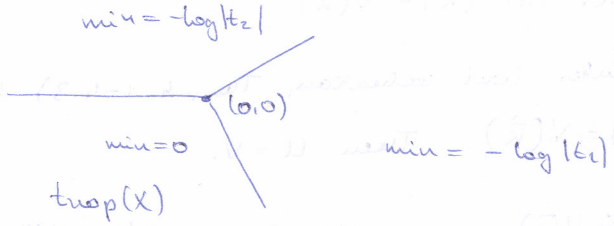
Then $\sum_i a_i x_i^{i_1} \cdots x_n^{i_n} = 0 \Rightarrow x \in R(f)$ by non-arch and Δ-regularity

Conv. let $(x_1, \dots, x_n) \in S$.

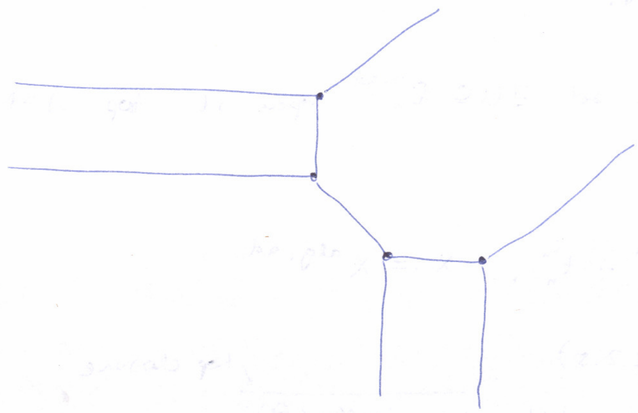
- Coord. trans. $\varphi_i \mapsto a_i t_i$, $a_i \in \bar{K}^\times$, $\log|a_i| = x_i$
  $\Rightarrow$ wma $x = (0, \dots, 0)$
- Change $\pi \in \bar{K}^\times$ s.t. $\log(\pi) = \tau_f(0) = \min_i(-\log|a_i|)$
  $\Rightarrow \pi \cdot f \in \mathcal{O}_K[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$
- We can find a solution $(\bar{\alpha}_1, \dots, \bar{\alpha}_n) \bmod m_K$ s.t. $\bar{\alpha}_i \neq 0 \ \forall i$

Since $\pi \cdot f$ is not a monomial, the lift $(a_1, \dots, a_n) \in V(f)(\bar{K})$ satisfies $\mathrm{trop}(\alpha_1, \dots, \alpha_n) = (0, \dots, 0)$

Ex. $X = V(t_1^{-1} + t_2^{-1} + 1) \subset \mathbb{C}_m^n$



$X = V(\pi^{-3} + \pi^{-2} t_1^{-1} + \pi^{-2} t_2^{-1} + \pi^{-3} t_1^{-1} t_2^{-1} + t_2^{-2} + t_1^{-2})$



VIII. The Bieri – Groves Theorem

Bieri – Groves: The geometry of valuations

We use additive notation today.

As usual: K, $|\cdot| : K \to \mathbb{R}_{\geq 0}$, $v := -\log|\cdot|$, $\Gamma := \log\sqrt{|K^\times|}$

<u>Recall</u>: Given $X/\mathrm{Spa}\,K$ loft, $f : X \to \mathbb{G}_m^n$ dominant, $f_1, \dots, f_n \in \mathcal{O}_X(X)^\times$

tropicalisation $\mathrm{trop}_f : X \to \mathbb{R}^n$, $x \mapsto (-\log|f_1(\tilde{x})|, \dots -\log|f_n(\tilde{x})|)$

§1 Statements

<u>Thm.</u> (BG) Let $f : X^{alg} \to \mathbb{G}_m^n$, X^{alg} oft $/\mathrm{Spa}\,K$, $X := X^{alg\,ad}$

Then $\mathrm{trop}_f(X)$ is a Γ-rational polyhedral set.

<u>Rmk.</u> We can deduce from this a statement about $X/\mathrm{Spa}\,K$ oft (i.e. qc & loft)

<u>Reformulation.</u> A K-alg oft, $G \subseteq A^\times$ f.g. subgroup, $G^* \simeq \mathrm{Hom}(G, \mathbb{R})\ (\simeq \mathbb{R}^n)$

Consider $\Delta_A^v(G) := \{\chi \in G^* \mid \exists w : A \to \mathbb{R} \cup \{\infty\}$ extending v s.t. $w|_G = \chi\} \subseteq G^*$

We can always assume G to be torsion-free: replace G by G^N.

$$\begin{tikzcd} K \arrow[d] \arrow[dr, "v"] & \\ A \arrow[r, "w"'] & \mathbb{R}\cup\{\infty\} \end{tikzcd}$$

Then if $G = \prod_{i=1}^n f_i^{\mathbb{Z}}$ then $\Delta_A^v(G) = \mathrm{trop}_f((\mathrm{Spec}\,A)^{ad})$

We do 2 further reduction steps.

<u>Lemma.</u>
- $\Delta_A^v(G) = \bigcup_{\substack{\mathfrak{q} \in A \\ \text{minimal prime}}} \Delta^v_{A/\mathfrak{q}}(G)$
- If A is an integral domain then $\Delta_A^v(G) = \Delta^v_{\mathrm{Frac}\,A}(G)$.

(This is not immediate since not every valuation of A passes to $\mathrm{Frac}\,A$.)

<u>Thm. A.</u> L/K field extn, $G \subseteq L^\times$ f.g. Then $\Delta_L^v(G)$ is a Γ-rational polyhedral set.

§2 Refinements

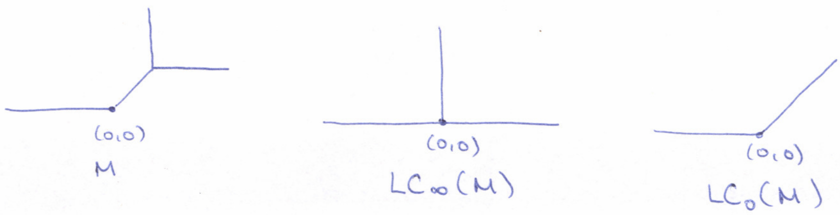
<u>Def.</u> $M \subseteq \mathbb{R}^n$ polyhedral set. The <u>local cone at ∞</u> is the polyhedral set

$LC_\infty(M) := \{x \in \mathbb{R}^n \mid \exists y \in M : y + \mathbb{R}_{\geq 0}\, x \subseteq M\}$.

The <u>local cone at 0</u> is $LC_0(M) := \bigcap_{\varepsilon > 0} \mathbb{R}_{\geq 0} \cdot (B_\varepsilon(0) \cap M)$

The <u>local cone at $x \in M$</u> is $LC_x(M) := LC_0(M - x) + x$

Ex.



Thm. C. $triv: K \to \{0\} \cup \{\infty\}$ trivial val. Then

- $LC_\infty(\Delta_A^v(G)) = \Delta_A^{triv}(G)$

- $LC_0(\Delta_A^v(G)) = \Delta^{triv}_{\mathcal{O}[G]/m[G]}(G)$ where $\mathcal{O} = K^\circ$, $\mathcal{O}[G]$ is the gen'd $\mathcal{O}$-algebra.

  $= \bigcup \Delta^{triv}_{k_i}(pr_i(G))$ where $\mathrm{Frac}\left(\mathcal{O}[G]/m\,\mathcal{O}[G] + \mathrm{Nil}\right) = \prod_i k_i$

  is the decomposition as a product of fields

Def. $M \subseteq \mathbb{R}^n$ polyhedral set is concave at $x \in M$ if

$$\text{Convex span}\,(LC_x(M)) = \text{Aff Lin Span}\,(LC_x(M))$$

M is totally concave if concave at $\forall x \in M$.

Thm. D. $\Delta_A^v(G)$ is totally concave.

§3 Overview of the proof

L/K f.g. field extn, $G \subseteq L^\times$

Wma G to be free, $n := rk_{\mathbb{Z}}\, G$

$m := trdeg\,(K(G)/K)$

Lecture 7: $n = m \iff \mathrm{Spec}\, A \longrightarrow \mathbb{C}_m^n$ dominant.

Then it follows that $\Delta_A^n(G) = \mathbb{R}^n$

Combining the two statements: 1) $trop_f\left((\mathrm{Spec}\, A)^{ad}\right) = \overline{trop_f\,(\mathrm{Spec}\, A)(\hat{K})}$ and

2) $x \in (\mathrm{Spec}\, A)(\hat{K})$ has a nbhd in $(\mathrm{Spec}\, A)^{ad}$ in which $trop_f$ is constant,

$$(trop_f)^{-1}(\gamma_1, \dots, \gamma_n) = (f^{ad})^{-1}\Big(\underbrace{trop\, id_{\mathbb{C}_m^n}(\gamma_1, \dots, \gamma_n)}_{\subset\, \mathbb{C}_m^{n\; ad} \text{ open affinoid}}\Big)$$

$\gamma_1, \dots, \gamma_n \in \Gamma^n$

Step 1. Case $n = m+1$

Step 2. General statements on polyhedral sets, see BG Chap. 4.

Step 3. General case.

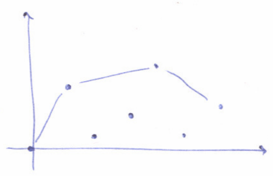
§4 Case $n = m+1$

Lemma. $k \xrightarrow{v} \mathbb{R} \cup \{\infty\}$ val field, $f \in k[t]$, $f = \sum_{i=0}^{n} a_i t^i$, $a_0 \neq 0$.

Then $\Delta^{v}_{k[t]/f}(\{t\}) =$ slopes of $\mathrm{Newt}(f)$.

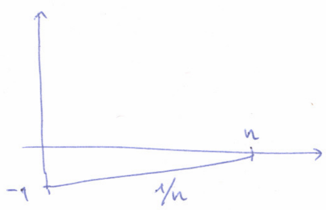
Recall. The Newton polygon of f is the upper convex hull of $\{(i, -v(a_i))\}$

Ex.



$t^n = p$

$k = \mathbb{Q}_p$



PF OF LEMMA: Some reductions first. $A := k[t]/f$, $g := \bar{t} \in A^\times$

- $\Delta^v_A(g) = \Delta^v_{A_{red}}(g)$ quotienting out by the nilradical doesn't alter valuations

  $= \bigcup_i \Delta^v_{k_i}(\mathrm{pr}_i(g))$ where $A_{red} = \prod_i k_i$.

- Omitted: $\mathrm{Slopes}(\mathrm{Newt}(f \cdot g)) = \mathrm{Slopes}(\mathrm{Newt}(f)) \cup \mathrm{Slopes}(\mathrm{Newt}(g))$

  $\Rightarrow$ Wma f to be irreducible.

- Extensions of valuations for purely inseparable extensions are unique.

  $\Rightarrow$ Wlog f separable.

- Fact. Extensions of valuations exist for any extension (pure abstract algebra, proof by Zorn's lemma)

- $L :=$ Gal closure of $k(g)$ in k^{alg}

  Then any extension from v to w for $k(g)$ extends to some $\tilde{w}$ on L

  Plus $\forall \sigma \in \mathrm{Aut}(L/K)$ $\forall \tilde{w}$: $\tilde{w} \circ \sigma$ is another valuation.

- Key Argument. $\tilde{w}$ on L any extension, $\alpha_1, \dots, \alpha_n \in L$ zeroes of f,

  $\{\lambda_1 < \dots < \lambda_r\} = \{\tilde{w}(\alpha_i)\}$ ordering, $n_i :=$ multiplicity of λ_i

  Then $$v(a_i) = \tilde{w}(a_i) = \tilde{w}\left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = n-i}} \prod_{j \in I} \alpha_j\right) = \begin{cases} n_1\lambda_1 + \dots + n_k\lambda_k & \text{if } n-i = n_1 + \dots + n_k \text{ for some } k \\ > n_1\lambda_1 + \dots + n_{k-1}\lambda_{k-1} + d\lambda_k & \text{if } n-i = n_1 + \dots + n_{k-1} + d,\ d \geq 1 \ \&\ d < n_k \end{cases}$$

Going back to the Newton polygon, we get that $\{\lambda_i\}$ are the slopes of $\mathrm{Newt}(f)$, each with multiplicity n_i. $\Rightarrow \Delta^v_{k(g)}(g^{\mathbb{Z}}) \subseteq \mathrm{Slopes}(\mathrm{Newt}(f))$.

$\mathrm{Aut}(L/K) \curvearrowright \{\alpha_i\}$ transitively $\Rightarrow$ $\supseteq$

Cor. Eisenstein Criterion.

We continue with the case $n = m+1$.

- Choose a basis $(g_1,\dots,g_n)$ for G.
- Any such basis also contains a tr. basis for $K(G)/K$, say $g_1,\dots,g_m$
- $f \in K(g_1,\dots,g_m)[t]$ minimal polynomial of g_{m+1}

$$f = \sum_{i\geq 0}\sum_{I\in\mathbb{N}_0} d_{I,i}\, g_1^{i_1}\cdots g_m^{i_m}\, t^i$$

Lemma. $\exists g\in G$: $G = H\times g^{\mathbb{Z}}$ where H is of rk m, $K(H)/K$ is of trdeg m and has g-minimal polynomial $f=\sum_{i=0}^{\ell} d_i h_i t^i$, $d_i\in K$, $h_i \in H$.

Pf: Set $g := g_{m+1}$, $H := \langle \tilde g_1,\dots,\tilde g_m\rangle$ where $\tilde g_i = g_i \cdot g_{m+1}^{-r_i}$ with $(r_1,\dots,r_m)\in\mathbb{N}^m$ not in the finite union of hyperplanes $\left\{\sum_{k=1}^m i_k r_k + i = \sum_{k=1}^m j_k r_k + j\right\}_{(I,i),(J,j)\in\mathbb{Z}^m\times\mathbb{Z}}$

where $d_{I,i}\neq 0\neq d_{J,j}$.

Then $$\tilde f = \sum_i \underbrace{\sum_I d_{I,i}\,\tilde g_1^{i_1}\cdots \tilde g_m^{i_m}}_{\neq 0 \text{ for at most one } I}\, t^{i+i_1r_1+\dots+i_mr_m}$$

Note that $\tilde f$ is still irreducible up to some factor t^d.

Claim. $\Delta^v_{K(G)}(G)$ is a Γ-rational polyhedral set.

Pf: • $K(H)\subseteq K(G)$ induces a surjection $\mathbb{R}^{m+1}\supseteq \Delta^v_{K(G)}(G) \twoheadrightarrow \Delta^v_{K(H)} \simeq \mathbb{R}^m$

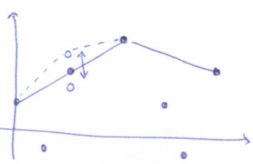
because any val. extends.

- Given $r=(r_1,\dots,r_m)\in\mathbb{R}^m$ corresponding to w_r, we get $\mathrm{Newt}_{w_r}(\tilde f)$ by applying w_r to the coeffs. Since the coeffs of $\tilde f$ are monomials, this only depends on r.

  By the Lemma from the finite extension case,
  $\pi^{-1}(r) = \{(r, \text{ slopes of } \mathrm{Newt}_r(\tilde f))\}$ is finite.

- NPI: $\mathbb{R}^m \longrightarrow \{0,1\}^{\{0,\dots,\ell\}}$

  $r \longmapsto \{i \text{ s.t. } (i,-v(\tilde a_i)) \text{ is on } \mathrm{Newt}_r(\tilde f)\}$ where $\tilde f = \sum_{i=0}^{\ell} \tilde a_i t^i$



The locus where NPI jumps is contained in $B :=$ Boundary $:= \{r \mid$ line through $(i_1,-v(\tilde a_{i_1}))$ and $(i_2,-v(\tilde a_{i_2}))$ is the line through $(i_3,-v(\tilde a_{i_3}))$, $(i_4,-v(\tilde a_{i_4}))$ for some $(i_1,i_2)\neq(i_3,i_4)\}$

Then $\mathbb{R}^m \setminus B = \coprod_i \mathring{C}_i$ for some m-dimensional polyhedra C_i Γ-rational

- On each $\mathring{C}_i$, NPI is constant. The slopes of $\mathrm{Newt}_r(\tilde{f})$, $r \in \mathring{C}_i$ can be computed by fin many aff lin Γ-rat functions $\mu_i^{(1)}, \dots, \mu_i^{(s_i)}$
- Continuity: for $r \in C_i \setminus \mathring{C}_i$, $\pi^{-1}(r) =$ all limit points of $\mu_i^{(j)}(\mathring{r})$ for $\mathring{r} \longrightarrow r$, $\mathring{r} \in \mathring{C}_i$

Conclusion: $\Delta^v_{K(G)}(G) = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s_i} \overline{\mu_i^{(j)}(\mathring{C}_i)}$ is a Γ-rat polyhedral set. $\square$

§5 General case

$m < n$ arbitrary

Lemma. $m \le t \le n$. Then $\{H \xhookrightarrow{\text{direct}} G \mid K(G)/K(H) \text{ finite}, H \text{ of } \mathrm{rk}\ t\} \subseteq Gr_{n,t}(\mathbb{Q})$ is dense in $Gr_{n,t}(\mathbb{R})$

End of PROOF: • $\mathcal{D} := \{H \xhookrightarrow{\text{dir}} G,\ \mathrm{rk}_{\mathbb{Z}} H = m+1,\ K(G)/K(H)\}$ is dense

- $\forall H \in \mathcal{D}$ $\mathrm{Im}\left(\Delta^v_{K(G)}(G) \longrightarrow \Delta^v_{K(H)}(H)\right) = \Delta^v_{K(H)}(H)$ is a Γ-rat polyh. set
- BG Thm 4.4 (gen. statement about polyhedral sets) $\Rightarrow$ $\Delta^v_{K(G)}(G)$ Γ-rat polyh. set. $\square$

09.11.20

Erratum to Lecture VI

1) (A, A^+) sheafy $\{\text{proj } \underline{\text{finite}} \text{ modules } / A\} \xrightarrow{\sim} \{\text{vb on } \mathrm{Spa}(A, A^+)\}$

2) $\mathcal{I} \subseteq \mathcal{O}_X$ qcoh sheaf of ideals

$Z(\mathcal{I}) = \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$ + $\mathcal{O}_X/\mathcal{I}$ as sheaf of rings + induced valuation

Correction: $Z(\mathcal{I})$ is adic if $X = \bigcup_i \mathrm{Spa}(A_i, A_i^+)$ where $\forall i$: A_i is either strongly noeth. OR has a noeth ring of defn.

[Huber's book §1.4]

In particular fine if $X/\mathrm{Spa}\, K$ lft or $X = \mathfrak{X}^{ad}$, $\mathfrak{X}$ noeth formal scheme

IX (p,q)-forms on polyhedral sets

[Gubler: Forms and currents on the analyfication of an alg vty]

§1 (p,q)-forms on $\mathbb{R}^n$

Coordinate-free approach: N fin free $\mathbb{Z}$-mod, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$,

$M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z}) \qquad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$

Recall: On $N_{\mathbb{R}}$ $\mathcal{A}^\cdot = \bigoplus_{p=1}^{n} \mathcal{A}^p$ sheaf of smooth diff forms,

$\mathcal{A}^p \cong \mathcal{C}^\infty \otimes_{\mathbb{Z}} \Lambda^p M = \{ f : N^{\times p} \longrightarrow \mathbb{C} \text{, alternating mult. linear} \}$

If $x_1, \dots, x_n$ $\mathbb{Z}$-basis of N, $dx_1, \dots, dx_n$ dual basis of M,

$$\mathcal{A}^p(U) = \Big\{ \sum_{i_1,\dots,i_p} \underbrace{f_{i_1,\dots,i_p}\, dx_{i_1} \wedge \dots \wedge dx_{i_p}}_{f_I\, dx_I,\ |I|=p} \ \Big|\ f_{i_1,\dots,i_p} \in \mathcal{C}^\infty(U) \Big\}$$

$\mathcal{A}^\cdot(U)$ graded algebra in $\wedge$, $\omega \wedge \eta = (-1)^{\deg \omega \deg \eta}\, \eta \wedge \omega$

$d : \mathcal{A}^p \longrightarrow \mathcal{A}^{p+1}$, $\displaystyle d(f_I dx_I) = \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$

Def. (p,q)-forms on $N_{\mathbb{R}}$: $\mathcal{A}^{p,q} := \mathcal{A}^p \otimes_{\mathcal{C}^\infty} \mathcal{A}^q$, $\displaystyle \mathcal{A}^{\cdot,\cdot} = \bigoplus_{0 \le p,q \le n} \mathcal{A}^{p,q} = \Lambda^\cdot (\mathcal{A}^p \oplus \mathcal{A}^q)$

Notation. $\omega \in \mathcal{A}^{p,q}(U)$, $\displaystyle \omega = \sum_{\substack{|I|=p \\ |J|=q}} f_{I,J} \underbrace{d'x_I \wedge d''x_J}_{(dx_{i_1},0)\wedge \dots \wedge (0, dx_{j_q})}$

Wedge product: $(f\, d'x_{I_1} \wedge d''x_{J_1}) \wedge (g\, d'x_{I_2} \wedge d''x_{J_2}) := fg\, dx_{I_1} \wedge d''x_{J_1} \wedge d'x_{I_2} \wedge d''x_{J_2}$
in $\Lambda^\cdot(\mathcal{A}' \oplus \mathcal{A}'')$. We get $\omega \wedge \eta = (-1)^{(p_1+q_1)(p_2+q_2)}\, \eta \wedge \omega$ where
ω is of type (p_1,q_1), η of type (p_2,q_2)

this is $p_1p_2 + q_1q_2$ in the paper of CL-D

Symmetry operator: $J(f\, d'x_I \wedge d''x_J) = f\, d''x_I \wedge d'x_J$
$= (-1)^{pq} f\, d'x_J \wedge d''x_I$

A (p,q)-form ω is symmetric if $J\omega = (-1)^p \omega$. Equivalently: $\forall I,J$: $f_{I,J} = f_{J,I}$.

Differentials: d': $\mathcal{A}^{p,q} \simeq \mathcal{A}^p \otimes_{\mathbb{Z}} \Lambda^q M \xrightarrow{d \otimes d} \mathcal{A}^{p+1} \otimes_{\mathbb{Z}} \Lambda^q M = \mathcal{A}^{p+1,q}$

$$d'(f\, d'x_I \wedge d''x_J) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J$$

Same for d'':

$$d''(f\, d'x_I \wedge d''x_J) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d''x_i \wedge d'x_I \wedge d''x_J = \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d'x_I \wedge d''x_i \wedge d''x_J \Big) (-1)^p$$

Pullback operator: N' free fin $\mathbb{Z}$-mod, $F : N' \longrightarrow N$.

F^*: $\mathcal{A}^{p,q}(U) \longrightarrow \mathcal{A}^{p,q}(F^{-1}(U)) \quad \forall U \subseteq N_{\mathbb{R}}$

$$F^*(f\, d'x_I \wedge d''x_J) = (f \circ F)\, \underbrace{F^*(d'x_I)}_{\text{pullback in } \mathcal{A}^{p,0}} \wedge \underbrace{F^*(d''x_J)}_{\text{pullback in } \mathcal{A}^{0,q}}$$

Ex. $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x,y) \mapsto xy$, t the variable in $\mathbb{R}$.

$$\left.\begin{aligned} & d'd''t = d'(d''t) = 0 \\ & d'(F^* d''t) = d'(y\, d''x + x\, d''y) = d'y \wedge d''x + d'x \wedge d''y \neq 0 \end{aligned}\right\} \; d' F^*_{\text{naive}} \neq F^*_{\text{naive}} d'$$

But: For $F: N'_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}$ affine linear, F^* commutes with d', d'', $d := d' + d''$.

Integration: $\mathcal{A}_c^{\cdot,\cdot}(U) :=$ forms with compact supp

$$\omega = f\, d'x_1 \wedge \ldots \wedge d'x_n \wedge d''x_1 \wedge \ldots \wedge d''x_n \in \mathcal{A}_c^{n,n}(U), \quad n = \mathrm{rk}_{\mathbb{Z}} N$$

$$\int \omega := \underbrace{(-1)^{\frac{n(n-1)}{2}}} \int f\, dx_1 \wedge \ldots \wedge dx_n$$

this sign comes from $\omega = (-1)^{\frac{n(n-1)}{2}} f\, d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_n \wedge d''x_n$.

For $F: N'_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}$ affine linear bijection: $\int F^*\omega = |\det \underbrace{\vec{F}}| \int \omega$

$\Rightarrow \int$ is canonical for $F \in \mathrm{Hom}_{\mathbb{Z}}(N', N) + N_{\mathbb{R}}$ — linear part of the aff lin map F

§2 Integration on polyhedra

$\Gamma \leq \mathbb{R}$ a $\mathbb{Q}$-sub-vector space

Def. σ Γ-rat polyhedron $\subseteq N_{\mathbb{R}}$ then $A_\sigma :=$ aff lin space spanned by σ,
$\mathbb{L}_\sigma :=$ corresponding subvector-space $\subseteq N_{\mathbb{R}}$

Note that since they are def'd by lin equations, $N_\sigma = N \cap \mathbb{L}_\sigma$ canonical set structure.

Def. $\dim \sigma := \dim \mathbb{L}_\sigma$.

Def. Let $\dim \sigma = d$. Then

$$\int_\sigma : \mathcal{A}_c^{d,d}(N_{\mathbb{R}}) \longrightarrow \mathbb{R}$$
$$\omega \longmapsto \int_{\sigma - x_0} \left[\mathbb{L}_\sigma \xrightarrow{+x_0} A_\sigma \hookrightarrow N_{\mathbb{R}}\right]^* \omega$$

where $x_0 \in \sigma$ is a chosen point.

Note. If $\sigma = \bigcap_i H_i$ then $A_\sigma \subseteq \bigcap_{\substack{i \\ \sigma \subseteq \partial H_i}} \partial H_i$

Rmk. $\mathrm{trop}(T^{ad})$ has canon. integral structure where $T/\mathrm{Spec}\, K$ split torus, i.e. $\exists\, T \simeq \mathbb{G}_m^n$.
Namely, $\mathrm{trop}(\mathbb{G}_m^{n,ad})$ has an integral structure $(\mathbb{Z}^n \subseteq \mathbb{R}^n) + \mathrm{Aut}(\mathbb{G}_m^n) = GL_n(\mathbb{Z})$.

Contraction: $\mathcal{A}^{p,q} = C^\infty \otimes_{\mathbb{R}} \Lambda^p M_{\mathbb{R}} \otimes_{\mathbb{R}} \Lambda^q M_{\mathbb{R}}$

$= \{$ multilin $f: N_{\mathbb{R}}^{\times p} \times N_{\mathbb{R}}^{\times q} \longrightarrow C^\infty$ alternating in first p and last q coordinates $\}$

Def. Given $v_1, \dots, v_s \in N_{\mathbb{R}}$, $I \subseteq \{1, \dots, p+q\}$, $\#|I| = s$, $\#|I \cap \{1, \dots, p\}| = s'$, $\#|I \cap \{p+1, \dots, p+q\}| = s''$, $\omega \in \mathcal{A}^{p,q}(U)$ let $\langle \omega, v_1, \dots, v_s \rangle_I := \omega(\dots v_1 \dots v_2 \dots v_s \dots) \in \mathcal{A}^{p-s', q-s''}(U)$

at indices I

E.g. $\langle f\, d'x_1 \wedge d''x_2,\ 3x_1 + 2x_2 \rangle_{\{1\}} = 3f\, d''x_2$

$\langle \quad \text{"} \quad , \quad \text{"} \quad \rangle_{\{2\}} = 2f\, d'x_1$

Def. • $\sigma \subseteq N_{\mathbb{R}}$ Γ-rat polyhedron. $\mathring{\sigma} :=$ relint σ = int of σ in $\mathbb{A}_\sigma$

- A face of $\sigma \subseteq N_{\mathbb{R}}$ is $\tau \subseteq \sigma$ s.t. $\tau = \sigma$ or $\tau = \sigma \cap H$ where H is a lin aff space s.t. $\mathring{\sigma} \cap H = \emptyset$. (A face of a Γ-rat polyhedron is a Γ-rat polyhedron.)
- For a face $\rho \subseteq \sigma$ of codim 1 we get $\mathbb{A}_\rho \subseteq \mathbb{A}_\sigma$, $\mathbb{L}_\rho \subseteq \mathbb{L}_\sigma$, $[n_{\rho,\sigma}] :=$ the generator of N_σ / N_ρ pointing outwards.
- Boundary integral: $\int_{\partial\sigma} : \mathcal{A}_c^{d-1,d}(N_\sigma) \longrightarrow \mathbb{R}$

$$\omega \longmapsto \sum_{\substack{\rho \subseteq \sigma \text{ codim } 1 \\ \text{face}}} \int_\rho \langle \omega, n_{\rho,\sigma} \rangle_{\{2d-1\}}$$

Similarly for $(d-1,d)$-forms.

Prop. (Stokes) σ d-dim Γ-rat polyhedron, $\omega \in \mathcal{A}_c^{d-1,d}(N_{\mathbb{R}})$.

$$\Rightarrow \int_\sigma d'\omega = \int_{\partial\sigma} \omega.$$

§3 (p,q)-forms on polyhedral complexes

Def. A (Γ-rational) polyhedral complex is a set $\mathcal{C}$ of Γ-rat polyhedra $\subseteq N_{\mathbb{R}}$ s.t. $\forall \sigma \in \mathcal{C}$ $\forall \rho \subseteq \sigma$ face: $\rho \in \mathcal{C}$.

Support: $|\mathcal{C}| = \bigcup_{\sigma \in \mathcal{C}} \sigma$

$\mathcal{A}_{\mathcal{C}}^{\bullet,\bullet}$ sheaf of superforms on $\mathcal{C}$: $U \longmapsto \{\omega \in \mathcal{A}^{\bullet,\bullet}(V) \mid V \subseteq N_{\mathbb{R}} \text{ open}, U = \mathcal{C} \cap V\} / \sim$

where $\omega \sim \omega' \Leftrightarrow \omega|_{\mathring{\sigma}} = \omega'|_{\mathring{\sigma}}$ $\forall \sigma \in \mathcal{C}$ polyhedron (resp $\sigma \subseteq |\mathcal{C}|$)

Def. $\mathcal{C}$ Γ-rat. polyh complex, pure of dim d

Weight on $\mathcal{C}$: $m: \mathcal{C}_d \longrightarrow \mathbb{Z}$, $\sigma \longmapsto m_\sigma$ where $\mathcal{C}_d := \{\sigma \in \mathcal{C} \mid \dim \sigma = d\}$

$$\int_{(\mathcal{C},m)} \omega := \sum_{\sigma \in \mathcal{C}_d} m_\sigma \int_\sigma \omega|_\sigma \quad \text{where } \omega \in \mathcal{A}_c^{d,d}(\mathcal{C})$$

$$\int_{\partial(\mathcal{C},m)} \omega := \sum_{\sigma\in\mathcal{C}_d} m_\sigma \int_{\partial\sigma} \omega \qquad \text{where} \qquad \omega \in \mathcal{A}_c^{d-1,d}(\mathcal{C})$$

Rmk. • $\nexists$ canon weight on $\bigcup_{r<d} \mathcal{C}_r = \partial\mathcal{C}$

• $\int_{\partial(\mathcal{C},m)} \omega$ doesn't depend just on $\omega|_{\partial\mathcal{C}}$

Prop. (Stokes) $\omega \in \mathcal{A}_c^{d-1,d}(N_\mathbb{R}) \Rightarrow \int_{(\mathcal{C},m)} d'\omega = \int_{\partial(\mathcal{C},m)} \omega$

§4 Tropical cycles

Def. $(\mathcal{C},m)$ weighted Γ-rat pure d-dim is balanced if $\forall \rho\in\mathcal{C}$, $\dim\rho = d-1$:
$\sum_{\substack{\sigma\in\mathcal{C}_d \\ \rho\subseteq\sigma}} m_\sigma n_{\rho,\sigma} \in N_\rho$. Then $(\mathcal{C},m)$ is a tropical cycle. (Or rather: sums of these.)

Thm. $X^{alg} \subseteq \mathbb{G}_{m,K}^n$ closed, pure of dim d, $X = X^{alg,ad}$.
K non-arch, complete, $|\cdot|$, $v = \log|\cdot|$
Then trop(X) is a tropical cycle.

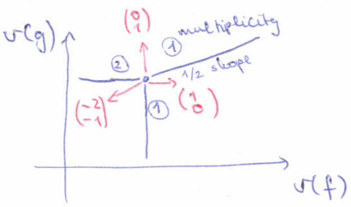
Idea. $X^{alg} \subseteq \mathbb{G}_m^2$ curve, embedding by $f,g \in \mathcal{O}(X^{alg})^\times$. Assume $\hat{K} = K$ for simplicity.
Let $x\in X$ be of type (2), i.e. $\mathcal{H}(x)/K$ has trdeg 1, $\Gamma_x = |K^\times| \cup \{0\}$
C_x/K curve def'd by $\mathcal{H}(x)$, $\overline{\{x\}} \simeq |C_x|$,
$f_x, g_x \in \mathcal{H}(x)^\times$ images of f,g
$\operatorname{div}(f_x) = \sum_{c\in|C_x|} a_c[c]$, $\operatorname{div}(g_x) = \sum_{c\in|C_x|} b_c[c]$

Then $\{$ rays originating from trop$(x) \subseteq \mathbb{R}^2\} = \{\mathbb{R}_{\geq 0}\cdot \binom{a_c}{b_c}\}_{c\in|C_x|}$

Multiplicity of $\mathbb{R}_{\geq0}\binom{a}{b} = \sum_{\substack{c \\ \mathbb{R}_{\geq0}\binom{a_c}{b_c} = \mathbb{R}_{\geq0}\binom{a}{b}}} \gcd(a_c,b_c)$

Balanced at x since $\deg(\operatorname{div}(f_x)) = 0 = \deg(\operatorname{div}(g_x))$

E.g. $\operatorname{div} f_x = 2c_1 - c_2 - c_3$, $\operatorname{div} g_x = c_1 - c_4$ $\qquad c_i \neq c_j \;\forall i\neq j$



$$\binom{-2}{-1} + 2\binom{1}{0} + \binom{0}{1} = 0$$

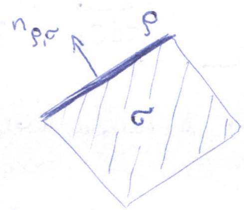
X More on tropical cycles

12.11.2018

N finite free $\mathbb{Z}$-mod, $N_{\mathbb{R}}$ of rk n.

$\mathcal{C} \subseteq N_{\mathbb{R}}$ polyhedral complex of pure dim d

Weight function $m: \mathcal{C}_d \to \mathbb{Z}$, $\sigma \mapsto m_\sigma$



$(\mathcal{C}, m)$ tropical cycle if balanced, i.e. $\forall \rho \in \mathcal{C}_{d-1}$: $\sum_{\substack{\sigma \in \mathcal{C}_d \\ \rho \subseteq \sigma}} m_\sigma \cdot n_{\rho,\sigma} \in N_\rho$

§1 (p,q)-currents

Def. The topology on $A_c^{p,q}(U)$, $U \subseteq N_{\mathbb{R}}$ open is the topology def'd by the following property. A net $(\omega_i)_{i\in I} \to \omega$ converges iff $\exists K \subseteq U$ cpt s.t. $\operatorname{supp} \omega_i \subseteq K$ $\forall i \in I$ and if $\omega_i = \sum_{I,J} f_{I,J}^{(i)} d'x_I \wedge d''x_J$, $\omega = \sum_{I,J} f_{I,J}\, d'x_I \wedge d''x_J$ then $\partial f_{I,J}^{(i)} \to \partial f_{I,J}$ $\forall I,J$ uniformly for every $\partial = \frac{\partial^{r_1} \partial^{r_2} \cdots \partial^{r_n}}{\partial x_1 \partial x_2 \cdots \partial x_n}$ differential form on U.

Def. A current is a continuous functional on $A_c^{\cdot,\cdot}(U)$.

$D^{\cdot,\cdot}(U)$ is the space of currents on U,

$D^{p,q}(U) := D_{n-p,n-q}(U) := \left(A_c^{n-p,n-q}(U)\right)^{\vee\, top}$

Def. Differentials d, d', d'' on $D^{\cdot,\cdot}$ are the duals of the corresponding differentials on $A_c^{\cdot,\cdot}(U)$ times $(-1)^{p+q+1}$.

Examples. 1) $A^{p,q}(U) \hookrightarrow D^{p,q}(U)$ (note that we don't have c anymore: the compactly supp are the test forms). S.o.w. currents are a generalisation of forms.

Here $\eta \mapsto [\eta]$ where $[\eta](\omega) = \int_U \eta \wedge \omega$ for $\omega \in A_c^{n-p,n-q}(U)$.

Then $[d'\eta] = d'[\eta]$ which motivates the sign $(-1)^{p+q+1}$.

2) Dirac current: $x \in U$, $\delta_x \in D^{n,n}(U)$, $\delta_x(f) = f(x)$.

Classically: for a smooth mf M, $f: M \to U$ proper (i.e. $f^{-1}(\text{cpt})$ is cpt) then $[M] \in D^{n-\dim M}(U)$, $[M](\omega) = \int_M f^*\omega$ for usual dim M-forms $\omega \in A^{\dim M}(U)$

3) $(\mathcal{C}, m)$ pure dim d polyhedral complex, weighted.

$\delta_{(\mathcal{C},m)} \in D^{n-d,n-d}(U)$, $\delta_{(\mathcal{C},m)}(\omega) = \int_{(\mathcal{C},m)} \omega \overset{\text{def}}{=} \sum_{\sigma \in \mathcal{C}_d} m_\sigma \int_\sigma \omega$

This generalises 2) to superforms.

Prop. (C,m) as above is balanced $\iff$ d'-closed, i.e. $d'\delta_{(C,m)}=0$.
$\iff$ d''-closed.

Rmk. Classically: if $M\subseteq U$ immersed, proper (i.e. closed img), no boundary then $d[M]=0$ by Stokes.

Thus (C,m) is the tropical analogue of a mf w/o bdry.

Pf OF PROP: By def: $\left(d'\delta_{(C,m)}\right)(\omega)=\delta_{(C,m)}(d'\omega)$ for $\omega\in A_c^{d-1,d-1}(U)$

By Stokes: $\delta_{(C,m)}(d'\omega)=\underbrace{\sum_{\sigma\in C_d} m_\sigma\int_\sigma d'\omega}_{\text{unnecessary intermediate step}} \underset{\text{Stokes}}{=} \sum_{\sigma\in C_d}\sum_{\substack{\rho\in C_{d-1}\\ \rho\leq\sigma}} m_\sigma\int_\rho\langle\omega, n_{\rho,\sigma}\rangle_{\{2d-1\}}$

$$=\sum_{\sigma\in C_d}\int_\rho\Big\langle\omega,\underbrace{\sum_{\rho\leq\sigma\in C_d} m_\sigma\cdot n_{\rho,\sigma}}\Big\rangle_{\{2d-1\}}$$

If (C,m) is balanced then $\forall\rho$: $\in N_\rho$ $\Rightarrow$ RHS$=0$.

Conversely, if (C,m) is not balanced at some ρ then look at $\mathring{\rho}$ and construct a $\underbrace{\text{bump form}}_{\text{form on a nbh of some } x\in\mathring{\rho}}$ s.t. RHS $\neq 0$. □

§2 Weights for tropicalisation

[Gubler, Guide to tropicalisation, Section 13]

$(K,|\cdot|)$ as usual, $v:=-\log|\cdot|$, $O=K^\circ$, $k=O/m$

$X\subseteq\mathbb{G}_m^n$ pure dim d

Recall: Bieri–Groves $\operatorname{trop}(X^{ad})$ is pure d-dim Γ-rat polyhedral set.

Def. Initial degeneration at 0: $\operatorname{in}_0(X)\subseteq\mathbb{G}_{m,k}^n$, $\operatorname{in}_0(X)=X^\circ\times_{\operatorname{Spec} O}\operatorname{Spec} k$

where $X^\circ:=$ flat closure of X in $\mathbb{G}^n_{m,O}$.

If $X=\operatorname{Spec} k[t_1^{\pm1},\dots,t_n^{\pm1}]/I$ then $X^\circ=\operatorname{Spec} O[t_1^{\pm1},\dots,t_n^{\pm1}]/\mathcal{J}$ where $\mathcal{J}=I\cap O[t_1^{\pm1},\dots,t_n^{\pm1}]$

(This X° is indeed O-flat since torsion-free)

Consider the maps to
- base change with $\tilde{k}/k$ any extn.
- translation by $\mathbb{G}_m^n(\tilde{k})$

} @

Def. Initial degeneration at $r\in\mathbb{R}^n$: $\operatorname{in}_r(X)$. Choose (L,v) extending (K,v) s.t. $r\in v(L)^n$, $r=v(\gamma_1,\dots,\gamma_n)$. Then $\operatorname{in}_r(X):=\operatorname{in}_0(\gamma^{-1}X)$. Well-def'd up to @.

Thm. (Sturmfels – Tevelev) $\quad \operatorname{trop}(X^{ad}) = \{ r \in \mathbb{R}^n \mid in_r(X) \neq \emptyset \}$

Ex. $X = V(f) \subseteq \mathbb{C}^n_{m,k}$, $\quad f = \sum_I a_I t_1^{i_1} \cdots t_n^{i_n}$

$\operatorname{trop}(X) = \{ r \in \mathbb{R}^n \mid \min_I \{ v(a_I) + r_1 i_1 + \dots + r_n i_n \}$ taken in several indices $\}$

(This was discussed in a previous lecture.)

Let $v(\pi) = \min_I v(a_i) \Rightarrow in_0(X) = V(\pi^{-1} f \bmod m) \subseteq \mathbb{C}^n_{m,k}$

In general: first perform a coordinate transformation $t_i \mapsto \gamma_i t_i$.

Def. $r \in \operatorname{trop}(X)$ is regular if $\exists \sigma \subseteq \operatorname{trop}(X)$ polyhedron of dim d s.t. $r \in \mathring{\sigma}$

The multiplicity at some regular point $r \in \mathbb{R}^n$ is

$$m(r, X) = \sum_{\substack{Z \subseteq in_r(X) \\ \text{irred. comp.}}} \underbrace{\operatorname{mult}(Z)}_{:= \operatorname{len}_{\mathcal{O}_{in_0(X), \eta_Z}} \left(\mathcal{O}_{in_0(X), \eta_Z} \right)}$$

Thm. (S – T) 1) $m(r, X)$ is locally constant on $\operatorname{trop}(X)_{reg}$.

2) Let C be any Γ-rational polyhedral complex str on $\operatorname{trop}(X)$.

Set $m: C_d \longrightarrow \mathbb{Z}$, $m_\sigma := m(r, X)$ $\forall r \in \mathring{\sigma}$. Then (C, m) is a tropical cycle.

Ex (continued). $r \in \operatorname{trop}(X)$ regular $\Leftrightarrow$ $\min_I$ taken in precisely 2 indices.

Initial degree of form $t_{i_1}^{m_1} \cdots t_{i_r}^{m_r} = \cancel{\alpha}\, t_{i_{r+1}}^{m_{r+1}} \cdots t_{i_n}^{m_n}$ with $\{i_j\}_j = \{1, \dots, n\}$, $m_j \geq 0$, $(\alpha \in k^*)$

E.g. $t_1^2 = t_2^2$ has mult 2 ,

$t_1^3 = t_2^2$ has mult 3 

Since we may perform coordinate transformations and assume $k = \bar{k}$, we actually don't need α.

Def. Cycle on $\mathbb{C}^n_m$: formal lin combinations of irred subvarieties: $\sum_{Z \subseteq \mathbb{C}^n_m} [n_Z] Z$

Tropical cycle on $N_\mathbb{R}$: formal lin combs of (C, m) balanced.

Rmk. If (C_1, m_1), (C_2, m_2) are d-dim balanced then we can also add "non-formally"*:

$(C_1, m_1) + (C_2, m_2) = (C_3, m_3)$ if $|C_3| = |C_1| \cup |C_2|$, $m_3 = m_1 + m_2$ on regular pts.

Then if $Z_1, Z_2 \subseteq \mathbb{C}^n_m$ are of pure d-dim, coprime and generically reduced then

$(\operatorname{trop}(Z_1 \cup Z_2), m) = (\operatorname{trop}(Z_1), m) + (\operatorname{trop}(Z_2), m)$

From the Thm. above, we get

$$\begin{aligned} \operatorname{Cyc}(\mathbb{C}^n_m) &\longrightarrow \operatorname{Cyc}(\mathbb{R}^n) \\ Z \text{ irred, reduced} &\longmapsto (\operatorname{trop} Z) \end{aligned}$$

*Note: $|C_3| \setminus |C_3|_{reg} = (|C_1| \setminus |C_1|_{reg}) \cup (|C_2| \setminus |C_2|_{reg})$.

§3 Pushforward on cycles

Let $\psi : \mathbb{C}_m^{n'} \longrightarrow \mathbb{C}_m^{n}$ a map of tori.

Then ψ is of the form $t_i \longmapsto \alpha_i t_1^{m_1^i} \cdots t_n^{m_{n'}^i}$

$\Rightarrow F := \mathrm{trop}(\psi) : \mathbb{R}^{n'} \longrightarrow \mathbb{R}^n$ Γ-rat. map

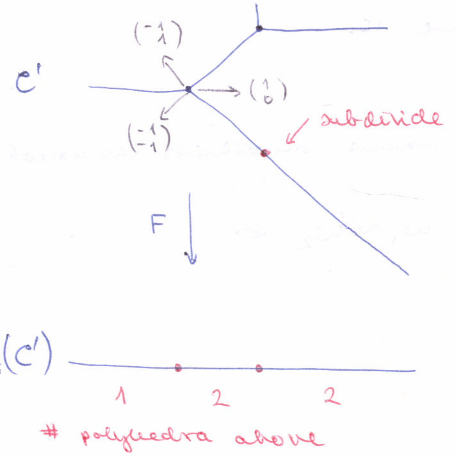
For $Z \subseteq \mathbb{C}_m^{n'}$ irr'ble: $\psi_*([Z]) := \begin{cases} 0 & \text{if } \dim \psi(Z) < \dim Z \\ [K(Z) : K(\psi(Z))] \cdot [\psi(Z)] & \text{otherwise} \end{cases}$

This defines $\mathrm{Cyc}(\mathbb{C}_m^{n'}) \longrightarrow \mathrm{Cyc}(\mathbb{C}_m^{n})$

Def. $(\mathcal{C}', m')$ Γ-rat polyhedral complex of dim $d \subseteq \mathbb{R}^{n'}$. Set

$F_*(\mathcal{C}') := \{ F(\rho') \mid \rho' \text{ face of some } \sigma' \in \mathcal{C}_d \text{ s.t. } F(\sigma') \text{ of dim } d \text{ again} \}$

Note that this need not be a polyh. complex in general but will become one if we subdivide $\mathcal{C}'$ further. Thus $F_*(\mathcal{C}')$ is a well-def'd polyh complex up to subdivision.



$$m_{F(\sigma')} := \sum_{\substack{\nu' \in \mathcal{C}' \\ \nu' \subseteq F^{-1}F(\sigma')}} m_{\nu'} \cdot [N_{F(\sigma')} : N_{\nu'}]$$

Prop. 1) $F_*(\mathcal{C}', m')$ is a tropical cycle if $(\mathcal{C}', m')$ is.

2) Projection formula: $\displaystyle\int_{F_*(\mathcal{C}', m')} \omega = \int_{(\mathcal{C}', m')} F^* \omega \quad \forall \omega \in A_c^{d,d}(U)$.

$\binom{-1}{-1} + \binom{-1}{1} + 2\binom{1}{0} = 0$ This shows that the converse of 1) need not hold.

PF OF PROP: 2) Integration is done polyhedron-wise $\Rightarrow$ suffices to check on a polyhedron. $\omega \in A_c^{d,d}(U)$, $\sigma' \in \mathcal{C}'_d$

Case 1. $\dim F(\sigma') < d$ $\Rightarrow$ $F|_{\sigma'}$ factors over $A_{F(\sigma')}$ with $\dim < d$ $\Rightarrow F^*\omega|_{\sigma'} = 0$

Case 2. $\dim F(\sigma') = d$ Consider $\vec{F} : N_{\sigma'} \longrightarrow N_\sigma$ which then has full rank

$|\det \vec{F}|_{N_\sigma}| = [N_\sigma : N_{\sigma'}]$ $\rightarrow \displaystyle\int_{\sigma'} F^*\omega = |\det \vec{F}|_{N_{\sigma'}}| \cdot \int_{F(\sigma')} \omega$. This implies 2). ✓

1) Assume $\delta_{(\mathcal{C}', m')}$ to be d'-closed. Then d' commutes with aff lin maps F.

$$\Rightarrow \int_{F_*(\mathcal{C}', m')} d'\omega = \int_{(\mathcal{C}', m')} F^* d'\omega = \int_{(\mathcal{C}', m')} d'(F^*\omega) \underset{\substack{\uparrow \\ d'\text{-closed}}}{=} 0$$

XI (p,q)-forms on algebraic varieties

Recall. $X \subseteq \mathbb{C}^n_m$ closed, pure d-dim

$C := \operatorname{trop}(X^{an})$ d-dim polyhedral set $\subseteq \mathbb{R}^n$

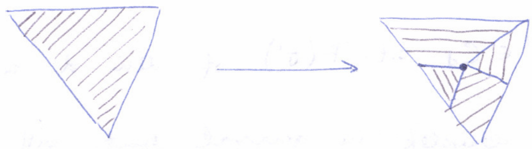
$\mathcal{C}$ polyhedral complex s.t. $|\mathcal{C}| = C$, this is a division of C

1) $x \in C$ is regular if $\exists\, \sigma \subseteq C$ polyhedron, $\dim \sigma = d$, $x \in \mathring{\sigma}$, $\mathring{\sigma}$ in a nbhd of x in C

Then $\bigcup_{\sigma \in \mathcal{C}_d} \mathring{\sigma} \subseteq C_{reg} := \{x \in C \text{ regular}\}$

Exc. TFAE: (1) $m : C_{reg} \longrightarrow \mathbb{Z}$ locally constant inducing balanced weights for every division $\mathcal{C}$

(2) $m : \mathcal{C}_d \longrightarrow \mathbb{Z}$ balanced weight for one division



This equivalence is intuitively clear.

Thm. (Sturmfels — Tevelev) $m : C_{reg} \longrightarrow \mathbb{Z}$, $x \longmapsto \sum_{\substack{W \in in_x X \\ \text{irred comp.}}} m_W$ is a balanced weight func.

This was supposed to clear up some of the confusion in lecture X.

2) C d-dim polyh set $\subseteq \mathbb{R}^n$,

$U \subseteq C$ open, $U = C \cap V$, $V \subseteq \mathbb{R}^n$ open

we implicitely assume smoothness (as usual)

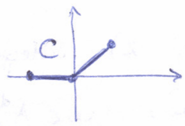
Def. A (p,q)-form on U is a (p,q)-form $\omega \in A^{p,q}(V)$ up to $\omega_1 \sim \omega_2 \Leftrightarrow$

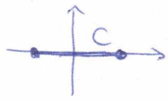
$\forall \sigma \in \mathcal{C}_d : \omega_1|_{\mathring{\sigma}} = \omega_2|_{\mathring{\sigma}}$ where $\mathcal{C}$ is any division of C.

In particular: $A^{p,q}(U) = 0$ if $\max(p,q) > d$.

Rmk. The definition is independent of V.

Rmk. The definition "depends on the embedding $C \hookrightarrow \mathbb{R}^n$"*.

E.g: $\alpha : \max\{0, x\} : [-1, +1] \longrightarrow \mathbb{R}$ is a smooth (0,0)-form with 

but not with 

* This is in quotation marks b/c we haven't specified in which category this should be understood.

§1 Tropical charts

$K, |\cdot|, v$ as usual

$X/\operatorname{Spec} K$ loc oft, purely d-dim, (separated)

$X^{ad} \xrightarrow{\pi} X^{an} = X_{max}$

Recall: $X^{an} = \{x \in X^{ad},\ \operatorname{rk} v_x = 1\}$ with the quotient topology,

$=$ the top space underlying the Berkovich space associated to X

Both X^{ad}, X^{an} are constructed locally on X then glued.

<u>Lemma</u>. $U = \operatorname{Spec} A \subseteq X$ open $\Rightarrow$ $U^{an} \overset{2)}{=} \{|\cdot| : A \to \mathbb{R}_{\geq 0}$ val extending $|\cdot|$ on $K\}$

with the topology such that all $\{|a(\cdot)|\}_{a \in A}$ are continuous;

$$U^{ad} \overset{1)}{=} \left\{ \begin{array}{ccc} v: A & \longrightarrow & \Gamma \cup \{0\} \\ \uparrow & & \uparrow \\ K & \xrightarrow{|\cdot|} & \operatorname{Im} |\cdot| \end{array} \text{ s.t. } v(\pi) \text{ is cofinal, i.e. } \pi \text{ top nilp wrt } K \text{ for some } \pi \in K \text{ uniformiser} \right\} / \cong$$

$U^{ad} \longrightarrow U^{an},\ x \longmapsto \tilde{x}$

Pf: 1) We have $A \longrightarrow \mathcal{O}_{U^{ad}}(U^{ad})$ from univ. prop.

Composing with $\mathcal{O}_{U^{ad}}(U^{ad}) \longrightarrow \mathcal{O}_{U^{ad},x} \xrightarrow{v_x} \Gamma_x \cup \{0\}$ gives a map LHS $\to$ RHS

Given $v \in$ RHS, let $A := K[t_1, \dots, t_n]/I$.

By the condition on v we have $\exists m: \pi^m t_1, \dots, \pi^m t_n$ are all top. nilp. wrt. v.

Then v factors via $K\langle \pi^m t_1, \dots, \pi^m t_n \rangle/(I)$, and hence occurs in $\underbrace{\mathbb{D}(0, |\pi|^m)^{\times n} \cap V(I)}_{\subseteq \mathbb{A}^{n,ad}}$

2) This is O.K. on sets, RHS of 2) is restriction of RHS of 1), rk 1-valuations with normalisation.

The $|a(\cdot)|$ are continuous by result of Huber (cf. VII)

To see that they define a topology, we can work in $\{|t_i(x)| < r\ \forall i\} \subseteq \mathbb{D}(0, |\pi|^{-m})^{\times n}$ for $m \longrightarrow 0$.

Then use that $A \longrightarrow \mathcal{O}_{U^{ad}}\left(\mathbb{D}(0, |\pi|^{-m})^{\times n} \cap U^{ad}\right)$ is dense and $\mathbb{D}(0, |\pi|^{-m})^{\times n} \cap U^{ad}$ is qc.

<u>Def</u>. (Gubler) A <u>tropical chart</u> on X is (U, φ, V) where

- $U \subseteq X$ open affine;
- $\varphi: U \hookrightarrow \mathbb{G}_m^n$ closed immersion;
- $V \subseteq U^{an}$ open of the form $\operatorname{trop}_\varphi^{-1}(\Omega)$ for some $\Omega \subseteq \mathbb{R}^n$ open.

A refinement of (U, φ, V) is some (U', φ', V') tropical chart with ψ s.t.

$U' \subseteq U$, $V' \subseteq V$ and

$$\begin{tikzcd} U' \arrow[r, "\varphi'"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{G}_m^{n'} \arrow[d, "\psi"] \\ U \arrow[r, "\varphi"] & \mathbb{G}_m^n \end{tikzcd}$$

Prop. 1) $\{V \subseteq X^{an} \mid \exists (U, \varphi, V) \text{ trop chart}\}$ form a basis of topology for X^{an}.

2) $(U, \varphi, V) \cap (U', \varphi', V') := (U \cap U', \varphi \times \varphi'|_{U \cap U'}, V \cap V')$ is a refinement of both trop charts (U, φ, V) and (U', φ', V') with refinement maps $\mathbb{C}_m^n \longleftarrow \mathbb{C}_m^{n+n'} \xrightarrow{pr} \mathbb{C}_m^{n'}$

(This is the first time we use separatedness of X.)

3) If (U, φ, V) is a trop chart, $U' \subseteq U$ affine, $V \subseteq (U')^{an}$ then $\exists \varphi', \psi$ s.t. (U', φ', V) is a trop chart.

Pf: 1) Local on X. Assume $X = \operatorname{Spec} A$, $\exists X \hookrightarrow \mathbb{C}_m^n$

X^{an} has basis $\underbrace{\{s_1 < |f_1(x)| < r_1, \dots, s_m < |f_m(x)| < r_m\}}_{\text{of the form}} =: V_{\underline{s} < \underline{f} < \underline{r}}$

Claim. $\{V_{\underline{s} < \underline{f} < \underline{r}} \mid \underline{r} = (r_1, \dots, r_m), \forall s_i > 0\}$ also forms a basis of X^{an}

Pf: $\{0 \leq |f(x)| < r\}$ are the ones not of this form.

Choose $\pi \in K^{\circ\circ}$, $|\pi| < r$. Then

$\{0 \leq |f(x)| < r\} = \{\frac{|\pi|}{2} < f(x) < r\} \cup \{\frac{\pi}{2} < |(f + \pi)(x)| < r\}$

If $V = V_{\underline{s} < \underline{f} < \underline{r}}$, $\underline{s} > 0$ then $V \subseteq D(f_1, \dots, f_m)$ and $(D(f_1, \dots, f_m), (\varphi, f_1, \dots, f_m), V)$ is a tropical chart. since $V = \operatorname{trop}_\varphi^{-1}\left(\mathbb{R}^n \times \prod_{i=1}^m (s_i, r_i)\right)$

2) $\Phi := (\varphi, \varphi') : U \cap U' \longrightarrow \mathbb{C}_m^{n+n'}$ closed embedding and $U \cap U'$ aff since X sep.

$\Rightarrow V \cap V' = \operatorname{trop}_\Phi^{-1}(\Omega \times \Omega')$ for the Ω, Ω' corresponding to V, V'.

3) Omitted.

Rmk. In Gubler's notes, $K = K^{alg}$ is assumed. Then for A/K integral reduced oft, $A^\times / K^\times$ fin gen free $\Rightarrow \operatorname{Spec} A \longrightarrow \operatorname{Spec} K[A^\times / K^\times]$ canonical

$t \longmapsto t$

Thus for φ, ψ we need extra datum.

§2 Differential forms on X^{an}

Def. (p,q)-form on an open $V \subseteq X^{an}$:

- $V = \bigcup_i V_i$ open covering with trop charts (U_i, φ_i, V_i)
- V_i: ω_i (p,q)-form on $\operatorname{trop}_{\varphi_i}(V_i)$ (this is open in $\operatorname{trop}_{\varphi_i}(U_i)$)

s.t. $\forall i,j$: $pr_i^* \omega_i|_{\operatorname{trop}_{\varphi_{ij}}(V_{ij})} = pr_j^* \omega_j|_{\operatorname{trop}_{\varphi_{ij}}(V_{ij})}$ wrt. $\mathbb{R}^{n_i} \longleftarrow \mathbb{R}^{n_i+n_j} \longrightarrow \mathbb{R}^{n_j}$

up to the obvious equivalence relation

$\{(U_i, \varphi_i, V_i, \omega_i)\} \sim \{(\tilde{U}_j, \tilde{\varphi}_j, \tilde{V}_j, \tilde{\omega}_j)\}$ if $\omega_i|_{V_i \cap \tilde{V}_j} = \tilde{\omega}_j|_{V_i \cap \tilde{V}_j}$ $\forall i,j$

Thus we get $\mathcal{A}^{p,q}$: sheaf of (p,q)-forms on X^{an}

Rmk. CL–D give local defn in terms of $X^{an} \supset V \xrightarrow{f_1,\dots,f_m} \mathbb{C}_m^{n,an}$ analytic moments

This defines the same sheaf of forms.

- On $\mathcal{A}^{\cdot\cdot} = \mathcal{A}^{\cdot\cdot}_{X^{an}}$ we have $d, d', d'', \wedge$, all defined on charts.

- Pullback: $X \xrightarrow{f} X/\operatorname{Spec} K$, $V \subseteq X^{an}$, $V' := f^{an,-1}V$

  $\omega \in \mathcal{A}^{p,q}(V)$ given by $(U_i, \varphi_i, V_i, \omega_i)$

  Then $f^*\omega$ is defined as follows:

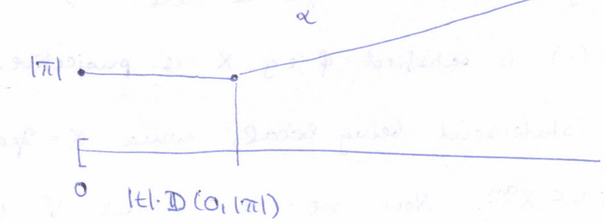
  $V' := \bigcup_j (U'_j, \varphi'_j, V'_j)$ with ψ_j refining $\{f^{-1}V_i\}$ s.t.
$$\begin{tikzcd} U'_j \arrow[r] \arrow[d, "\varphi'_j"] & U_{i(j)} \arrow[d, "\varphi_{i(j)}"] \\ \mathbb{C}_m^{n'_j} \arrow[r, "\psi_j"'] \arrow[ur, phantom, "G"] & \mathbb{C}_m^{n_{i(j)}} \end{tikzcd}$$

  $f^*\omega$ is given by $\{\psi_j^* \omega_{i(j)} |_{\operatorname{trop}_{\varphi'_j}(V'_j)}\}$

§3 Comparison with the notion from VII

$X = \mathbb{A}^1_K$, $X^{an} \xrightarrow{|t|} \mathbb{R}_{\geq 0}$ (i.e. this is the absolute value fn. on the coordinate t)

$x \longmapsto d(x,0)$



$\alpha := \max\{|\pi|, x\} \circ |t|$

$\alpha: \mathbb{A}^1_K \longrightarrow \mathbb{R}$ continuous

$$\left.\begin{array}{l} \alpha|_{D(0,|\pi|)} = |\pi| \\ \alpha|_{A(|\pi|,\infty)} = |t| \end{array}\right\} \text{adic locally, } \alpha \text{ is given as } |f|,\ f \text{ unit}$$

But α is not smooth in the sense of our defn of $(0,0)$-forms.

Reason: $\mathbb{D}(0,|\pi|) \cup A(|\pi|,\infty)$ is not the inverse img of a covering of $\mathbb{A}^{1,an}$

Namely, $\{\pi^{-1}S,\ S \subseteq \mathbb{A}^{1,an}\} = \{T \subseteq \mathbb{A}^{1,ad}$ closed under specialisation & generalisation$\}$

$\mathbb{A}^{1,an}$ quot top wrt π $\Rightarrow$ $U \subseteq \mathbb{A}^{1,ad}$ open of form $\pi^{-1}U'$, U' open

$\Leftrightarrow$ U closed under gen & spec.

$\mathbb{D}(0,|\pi|) \ni \eta_{0,|\pi|}$, $\overline{\{\eta_{0,|\pi|}\}} \ni \eta^{\infty}_{0,|\pi|} \notin \mathbb{D}(0,|\pi|)$

19.11.20

Erratum: Def. $C \subseteq \mathbb{R}^n$ polyh complex if stable under faces and $\forall \sigma \neq \rho$: $\mathring{\sigma} \cap \mathring{\rho} = \emptyset$

XII (p,q)-forms on alg varieties 2

$K, |\cdot|, v$ as usual

X/K loft separated

We have $X^{ad} \longrightarrow X^{an}$ and for $V \subseteq X^{an}$ open, $\omega \in A^{p,q}(V)$ given by $(U_i \varphi_i, V_i, \omega_i)$

§1 Properties of X^{an}

<u>Prop.</u> X^{an} is 1) T2,

2) locally compact: $\forall x \in V \subseteq X^{an}$ $\exists V^c \subseteq V$ compact nbhd

3) locally path-connected, and path-connected when X is

Pf. 1) Only in the special case$^{(*)}$ when $\forall x, y \in X$ $\exists U \subseteq X$ open affine, $x, y \in U$.

Let $x, y \in X^{an}$. Get $\operatorname{Spec} \kappa(x), \operatorname{Spec} \kappa(y) \longrightarrow X$. Choose U for these two points.

$\Rightarrow x, y \in U^{an}$.

XI: if $U = \operatorname{Spec} A$ then $U^{an} = \{ |\cdot| : A \longrightarrow \mathbb{R}_{\geq 0}$ extending $|\cdot|$ on $K \}$

$x \neq y \Rightarrow \exists a \in A: \; |a(x)| \neq |a(y)|$

Then for $(a_1, b_1), (a_2, b_2)$ suitable disjoint: $x \in |a|^{-1}(a_1, b_1)$, $y \in |a|^{-1}(a_2, b_2)$, showing T2 in this special case. ✓

<u>Rmk.</u> (*) is satisfied if e.g. X is projective.

2) The statement being local, wma $X = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} K[t_1, \ldots, t_n]$.

Let $x \in V \subset X^{an}$. Now we can shrink V to $x \in V' = \bigcap_{fin} \{ r_i < |f_i| < s_i \}$, $f_i \in A$

XI: wma $\forall r_i > 0$.

Pick $r_i < r_i' < |f_i| < s_i' < s_i$ where $r_i', s_i' \in \sqrt{\operatorname{Im} |K^\times|}$

Then $x \in \bigcap_i \left(\{ |f_i| \leq s_i' - \varepsilon \} \cap \{ r_i' + \varepsilon \leq |f_i| \neq 0 \} \right) \cap \bigcap_{\text{in } A^{n,ad}} \mathbb{D}(0, \varepsilon^{-1})^{\times n}$ for ε small rational.

This is affinoid U in X^{ad}

Then $q(U) \subseteq X^{an}$ is a qc nbhd of x where $q: X^{ad} \longrightarrow X^{an}$.

3) Omitted, see [Berkovich]. □

§2 Comparison of "smooth" and "smooth on X^{ad}"

<u>Def.</u> $\psi: \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>piecewise smooth</u> if

- ψ is continous
- $\exists C$ Γ-rational polyhedral complex, $|C| \subseteq \mathbb{R}^n$, $\forall \sigma \in C: \psi|_\sigma$ is smooth.

<u>Def.</u> $f: X^{an} \longrightarrow \mathbb{R}$ is <u>piecewise smooth</u> if

- f cont.
- $\exists$ covering of X^{an} by tropical charts in which f is given by pw. smooth fuctions.

Prop: (Gubler – Künnemann, §8) $f : X^{an} \longrightarrow \mathbb{R}$ continuous. TFAE:

(1) f is pw. smooth

(2) $\exists X^{ad} = U_i$ open covering and $\varphi_i : U_i \longrightarrow C_m^{n,ad}$ and ψ_i smooth on $\mathbb{R}^n$ s.t.

$f|_{U_i} = \psi_i \circ \mathrm{trop}_{\varphi_i}$

Pf: (1) $\Rightarrow$ (2): Let $V \subseteq X^{an}$ be open with tropical chart (U, φ, V) s.t.

$f|_V = \psi \circ \mathrm{trop}_\varphi$ for ψ pw. smooth function.

Let $V^{ad} := q^{-1}(V)$.

Let C be a complex of definition for ψ.

Claim. $V^{ad} \subseteq \bigcup_{\sigma \in C} (\mathrm{trop}_\varphi \circ q)^{-1}(\sigma)^\circ$

From this (1) $\Rightarrow$ (2) follows since $f|_{(\mathrm{trop}_\varphi \circ q)^{-1}(\sigma)^\circ}$ is given by smooth factorisation.

PF OF CLAIM: Enough to show $C_m^{n,ad} = \bigcup_{\sigma \in C} \mathrm{trop}^{-1}(\sigma)^\circ$

• Since $\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ$, we may refine C.

• Also assume that C is the smallest complex refining all the complexes $\{\{ l \geq \gamma\} \cup \{l = \gamma\} \cup \{l \leq \gamma\}\}$

for $\{(l, \gamma)\}$ occuring in the def of some $\sigma \in C$.

• Since $A^\circ \cap B^\circ = (A \cap B)^\circ$ and any $\sigma \in C$ is the intersection of half-spaces $C_{l,\gamma}$, it is enough

to check for $C_{l,\gamma}$.

• (l, γ) corresponds to $|f_1^{l_1} \cdots f_n^{l_n}| \lesseqgtr \gamma$ and clearly $C_m^{n,ad} = \{\prod |f_i^{l_i}| > \gamma\} \cup$

This proves the Claim. $\cup \{\prod |f_i^{l_i}| \leq \gamma\}$

(2) $\Rightarrow$ (1) (SKETCH) Let $f : V^{an} \longrightarrow \mathbb{R}$ be of type 2. Local $\Rightarrow$ wma X affine,

$x \in V \subseteq X^{an}$, $X = \mathrm{Spec}\, A$.

Then $\overline{\{x\}}^{V^{ad}}$ is qc and $\simeq \mathrm{Spa}(\mathcal{H}(x), \mathcal{O}_K/m_K)$ since $V^{ad} = q^{-1} V$ is partially proper.

Let $\overline{\{x\}} \supseteq \bigcup_i U_i$ be a finite affine covering and $\varphi_i : U_i \longrightarrow C_m^{n,ad}$ s.t. $f|_{U_i} = \psi_i \circ \mathrm{trop}_{\varphi_i}$,

where ψ_i is smooth.

Idea 1. $\varphi_i = (\varphi_{ij})_j$

Since A is dense in $\mathcal{O}(U_i)$ and U_i are qc, we may approximate φ_{ij}

by $\tilde{\varphi}_{ij} \in A$ s.t. $\mathrm{trop}_{\tilde{\varphi}_{ij}}|_{U_i} = \mathrm{trop}_{\varphi_{ij}}$.

Assuming also $x \in U_i, V_i$ we get $\tilde{\varphi}_{ij}(x) \neq 0$.

Shrinking the U_i, wma $\bigcup_i U_i \subseteq D\left(\prod_{i,j} \widetilde{\varphi_{ij}}\right)^{ad}$

We get that $(\widetilde{\varphi_{ij}})_{ij}: \quad D\left(\prod \widetilde{\varphi_{i,j}}\right)^{ad} \longrightarrow \mathbb{C}_m^{\sum_i n_i}$ define tropical chart for all U_i.

In a refined construction, f expresses as a pw. smooth function in this.

Idea 2. Prop. If $U \subseteq X^{ad}$ is open and $x \in q(U)$ s.t. $q^{-1}(x) \subseteq U$ then $q(U)$ is a nbhd of x.

Pf: Wma U qc using that the fibre $q^{-1}(x) = \overline{\{x\}}$ is qc.

Then $q(U)$ is closed since X^{an} is T2. Thus $q^{-1}(q(U))$ is closed by continuity of the quotient. Hence $q^{-1}(q(U))$ is contained in a large affinoid, and as such, also qc.

Then $\partial U = q^{-1}(q(U)) \setminus U$ is qc. $\Rightarrow$ $q(\partial U)$ is qc and $x \notin q(\partial U)$ by assumption.

X^{an} T2 $\Rightarrow$ $\exists V$ open, $x \in V$, $V \cap q(\partial U) = \emptyset$.

Then $q^{-1}(V) \cap \partial U$ is open and closed under spec./gen. $\Rightarrow$ $q(q^{-1}V \cap U)$ is an open nbhd of x.

§3 Dimension

X adic space, loft $/\mathrm{Spa}\, K$

Def. 1) Krull dimension: $\dim X := \sup\{n \mid \exists$ chain of specialisations $x_0 \rightsquigarrow x_1 \rightsquigarrow \dots \rightsquigarrow x_n$, $x_i \neq x_{i+1}\ \forall i\}$

2) Transcendence dimension.

For L non-archimedean, $E \subseteq L$ is alg-top-dense if $E^0 = E$, $E^{i+1} = \overline{E^{i,alg}}^{top}$ and $E^n = L$ for some $n \gg 0$.

For $x \in X$: $\operatorname{trdim}(\kappa(x) \mid K) := \inf\{n \mid \exists E/K$ of trdeg n s.t. $E \subseteq \kappa(x)$ alg-top-dense$\}$

3) Berkovich dimension: $x \in X$ max rk 1 point, $s(x) := \operatorname{trdeg}\left(\widetilde{\kappa}(x), \mathcal{O}_K/m_K\right)$

Then $t(x) := \operatorname{rk}_{\mathbb{Z}}\left(\operatorname{Im} |\cdot(x)| / |K^\times|\right)$ and $d(x) := t(x) + s(x)$.

$\text{Berk-dim}(X^{an}) := \sup\{d(x) \mid x \in X^{an}\}$

Prop. $X/\mathrm{Spa}\, K$ loft. 1) Then $\dim X = \operatorname{trdim} X = \text{Berk-dim}\, X$

2) If X is oft then these are all finite.

3) If $X = (X^{alg})^{ad}$ for $X^{alg}/\mathrm{Spec}\, K$ oft d-dimensional then $\dim X = d$.

§4 Support of (p,q)-forms

Same setup as in §1–2.

Def. $V \subseteq X^{an}$, $\omega \in \mathcal{A}^{p,q}(V)$. The support of ω is

$$V \setminus \{x \in V \mid \exists\, x \in U \subseteq V \text{ open s.t. } \omega|_U = 0\} =: \operatorname{supp} \omega$$

Note that $\operatorname{supp} \omega \subseteq V$ is closed by def.

Prop. 1) X pure d-dim. Then $\mathcal{A}^{p,q} = 0$ when $\max(p,q) > d$.

2) More precisely, let $\omega \in \mathcal{A}^{p,q}(V)$, $x \in V$. Then $\max(p,q) > d(x) \Rightarrow x \notin \operatorname{supp}\omega$.

Rmk. 2) generalises the fact that $|f|$ is constant near classical points $\forall f \in \mathcal{O}^\times$.

PF OF PROP.: 1) is clear since $\forall U \subseteq X$ aff open, $\forall \varphi : U \hookrightarrow \mathbb{G}_m^n$: $\operatorname{trop}_\varphi(U)$ is a d-dimensional polyhedral set.

2) Prop. (U, φ, V) trop. chart, $x \in V$. $\Rightarrow \exists\, x \in W \subseteq V$ compact s.t. $\operatorname{trop}_\varphi(W)$ is a d-dim polyhedral set. This clearly implies 2).

Pf.: Consider $\overline{\{x\}}^{ad} \subseteq X^{ad}$. For each $y \in \overline{\{x\}}$ minimal (i.e. no specializations):

$$\operatorname{rk}_{\mathbb{Z}} \, {}^{\operatorname{Im} v_y}\!/_{\operatorname{Im} v_y|_{K^\times}} \leq d(x).$$

Taking all relations of the $v_y(\varphi_{i,y})$ in this group (where $\varphi = (\varphi_i)$), they define locally a rational affinoid (in part, open) $U_y \subseteq X^{ad}$

Also $\operatorname{trop}_\varphi(U_y) \subseteq$ d-dim subspace of $\mathbb{R}^n$.

Using $\overline{\{x\}}$ qc again: $\exists\, y_1, \dots, y_r$, $\overline{\{x\}} \subseteq \bigcup_{j=1}^r U_{y_j}$

By the Prop. (i.e. Idea 2) from §2, $q\left(\bigcup_{j=1}^r U_{y_j}\right) \subseteq X^{an}$ is a qc nbhd of x.

§5 Integration

X pure d-dim

Cor. $U \subseteq X$ open dense, $x \in X^{an}$ s.t. $d(x) = \dim X$. Then $x \in U^{an}$.

Pf: $\dim(X \setminus U) < d \Rightarrow \forall y \in X^{an} \setminus U^{an}$: $d(y) < d$

Def. $\omega \in \mathcal{A}^{p,q}(V)$ has compact support if $\operatorname{supp}\omega$ is compact.

$\mathcal{A}^{p,q}_c(V) := \{\omega \in \mathcal{A}^{p,q}(V) \mid \operatorname{supp}\omega \text{ is compact}\}$

Note. If $V \subseteq W$ are opens then we get $\mathcal{A}^{p,q}_c(V) \longrightarrow \mathcal{A}^{p,q}_c(W)$.

Cor. Let $\omega \in \mathcal{A}_c^{p,q}(V)$, $\max(p,q) = d$. Then $\exists\, U \subseteq X$ aff open and $\varphi : U \longrightarrow \mathbb{G}_m^n$ moment s.t. $\operatorname{supp}\omega \subseteq U^{an}$ and ω is rep'd by some $\mathcal{A}_c^{p,q}(\operatorname{trop}_\varphi(U))$.

Pf: Any open dense $U \subseteq X$ is s.t. $\operatorname{supp}\omega \subseteq U^{an}$ $\Rightarrow$ since X is irr'ble.

If ω is given by fin many $(U_i, \varphi_i, V_i, \omega_i)$ then $\bigcap_i U_i$ does the job. $\square$

Def. $\omega \in \mathcal{A}_c^{p,q}(V) \subseteq \mathcal{A}_c^{d,d}(X^{an})$. Then $\displaystyle\int_V \omega := \int_{\operatorname{trop}(U)} \omega_U$ where (U, ω_U) is as in the previous Cor. We call (U, ω_U) a chart of integration.

Exc. 1) $\int_V \omega$ is indep of (U, ω_U)

2) $\displaystyle\int_V \lambda\omega + \mu\eta = \lambda \int_V \omega + \mu \int_V \eta.$

23.11.2018

XIII Geometry of curves

§1 Motivation

$(K, |\cdot|, v)$ as usual, $\mathcal{O}_K = K^\circ$, $k = \mathcal{O}_K / m_K$

$X^{alg}/\operatorname{Spec} K$ smooth projective curve, associate $X^{ad} \longrightarrow X^{an}$

Given a "nice" model $\mathfrak{X}/\operatorname{Spec}\mathcal{O}_K$: flat proper scheme over $\operatorname{Spec}\mathcal{O}_K$ with $\mathfrak{X} \otimes K \simeq X^{alg}$

we can consider $\mathfrak{X} \otimes_{\mathcal{O}_K} k$ proper curve over $\operatorname{Spec} k$

Attach a graph with vertices $:= \{$irred components of $\mathfrak{X} \otimes k\}$

edges $:= \{$intersection points of the irreducible components$\}$

Nice := semistable := the irred components are reduced and singularities are ordinary double points

$\hat{\mathcal{O}}_{\mathfrak{X}\otimes k, x} \simeq k[[X, Y]]/XY$

Strongly semistable := irred components are smooth, and $\mathfrak{X}$ is semistable

Then $\exists$ canonical continuous $\Sigma(\mathfrak{X}) \rightleftarrows X^{an}$ and a retraction

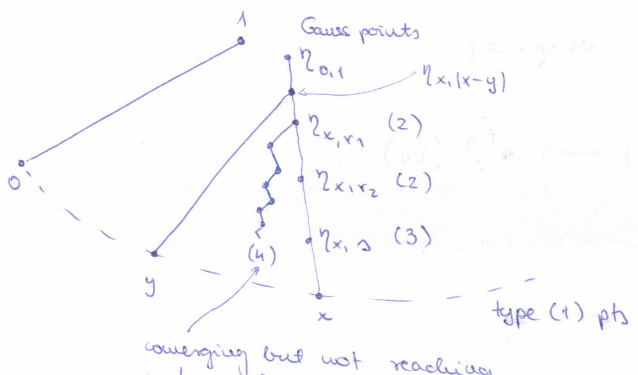
It turns out that any type (2) or (3) point occurs in $\Sigma(\mathfrak{X})$ for some model $\mathfrak{X}$.

$\rightarrow$ gives $\{(2), (3) \in X^{an}\}$ structure of "infinite graph"

Then $\mathcal{A}^{\cdot,\cdot}$ can be described by this plus retraction maps

Example of $\mathbb{P}^1$: $\mathbb{P}^{1,an}$ looks like $\{\infty\} \cup \bigcup_n \mathbb{D}(0, r^{-n})^{an}$

$\mathbb{D}(0,1)^{an}$:

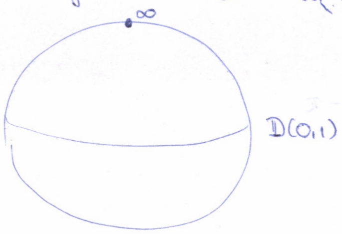


$r_1 > r_2 \in |K^\times|$

$s \notin |K^\times|$

In particular: given $x_1, x_2 \in \mathbb{D}(0,1)^{an}$ $\exists !$ path $[x_1, x_2]$

$\Rightarrow$ for $\mathbb{D}(0,1)^{an}$, the graph is tree-like. We can also put a metric on this infinite tree: $d(\eta_{x,r_1}, \eta_{x,r_2}) := \log r_2 - \log r_1$



§ 2 Admissible models

Def. A formal scheme $\mathfrak{X}/\mathrm{Spf}\, \mathcal{O}_K$ is admissible if can be covered by Spf A's where $A \simeq \mathcal{O}_K\langle t_1, \dots, t_n\rangle / \mathfrak{a}$ where

- $\mathcal{O}_K\langle t_1, \dots, t_n\rangle$ restricted power series over $\mathcal{O}_K$, i.e. π-adic completion of $\mathcal{O}_K[t_1, \dots, t_n]$ where $\pi \in K^{oo} \setminus \{0\}$
- $\mathfrak{a}$ is a fin gen ideal
- A is $\mathcal{O}_K$-flat (i.e. π-tors free)
- A has π-adic top.

In short: flat, π-adic and "formally of finite presentation over $\mathrm{Spf}\, \mathcal{O}_K$".

Eg. $(\mathbb{A}^n_{\mathcal{O}_K})^\wedge_\pi$ but not $\mathrm{Spf}\, \mathcal{O}_K[\![t]\!]$

Def. $\mathfrak{X}' \longrightarrow \mathfrak{X}$ is an admissible blowup if of the form $\varprojlim \left(\mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{J}^m \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n\right)$ where $\mathcal{J}$ is an open coherent ideal.

Equivalent local description: if $\mathfrak{X} = \mathrm{Spf}\, A$ then $\mathfrak{X}' = \pi$-adic completion of blowup of Spec A in some open coherent ideal I

Facts. • Completion of admissible blowup is admissible blowup.

• $\{$adm blowup $\mathfrak{X}' \to \mathfrak{X}\}$ is filtered, i.e. $\exists\, \mathfrak{X}'''$

$$\begin{tikzcd} & \mathfrak{X}' \arrow[dr, "\forall"] & \\ \mathfrak{X}''' \arrow[ur, dashed] \arrow[dr, dashed] & & \mathfrak{X} \\ & \mathfrak{X}'' \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, dashed] \arrow[d, dashed, "\mathrm{abu}"'] & \mathfrak{Y} \arrow[d, "\forall \mathrm{abu}"] \\ \mathfrak{X} \arrow[r, "\forall"] & \mathfrak{Y} \end{tikzcd}$$

Consequence. We can localise $(\mathrm{Adm\,Formal}/\mathrm{Spf}\, \mathcal{O}_K)[\mathrm{Adm\,Blup}^{-1}]$

Fact. • An abu $\mathfrak{X}' \to \mathfrak{X}$ induces $(\mathfrak{X}')^{ad}_\eta \xrightarrow{\simeq} \mathfrak{X}^{ad}_\eta$

Thm. (Raynaud). The natural functor

$$\{\text{qcqs adm formal } \mathfrak{X}/\mathrm{Spf}\, \mathcal{O}_K\}[\mathrm{Adm\,Blup}^{-1}] \longrightarrow \{\text{qs tft } X/\mathrm{Spa}\, K\}$$

is an equivalence of categories.

In ptic, any $X/\mathrm{Spa}\, K$ qs tft has an adm formal model.

Moreover, the specialisation maps induces $(X, \mathcal{O}_X^+) \xrightarrow{\simeq} \varprojlim_{\mathfrak{X}} (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ iso in LRS.

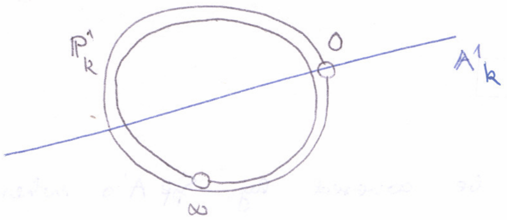
§3 Examples for curves

An admissible blowup.

$$\mathfrak{X} = \left(\mathbb{A}^1_{\mathcal{O}_K}\right)^\wedge_\pi = \mathrm{Spf}\ \mathcal{O}_K\langle t\rangle, \quad X = \mathbb{D}(0,1), \quad \mathfrak{X}' = \left(BL_{(\pi,t)}\, \mathbb{A}^1_{\mathcal{O}_K}\right)^\wedge_\pi$$

We get 2 charts: $\quad U := \mathrm{Spec}\ \mathcal{O}_K[t,u]/_{ut-\pi}, \quad V := \mathrm{Spec}\ \mathcal{O}_K[t,v]/_{v\pi - t}$

Reductions $\otimes k$: $\quad \mathrm{Spec}\ k[t,u]/_{tu} \qquad \mathrm{Spec}\ k[v]$



$$\mathrm{sp}: |\mathbb{D}| \longrightarrow |\mathfrak{X}'| \quad \begin{cases} A(1,1) \longrightarrow \mathrm{Spec}\ k[t,t^{-1}] & \text{as usual} \\ \mathring{A}(\pi,1) \longmapsto 0 & \\ A(\pi,\pi) \longrightarrow \mathrm{Spec}\ k[u,u^{-1}] & \\ \quad x \longmapsto \mathrm{sp}(x/\pi) & \\ \mathring{\mathbb{D}}(0,\pi) \longmapsto \infty & \end{cases}$$

Characterisation of (2) and (5) points in $|X| \simeq \varprojlim_{\mathfrak{X}} |\mathfrak{X}|$

X curve

Type (2): comp systems that are eventually the generic point of an irreducible component of some $\mathfrak{X} \otimes_{\mathcal{O}_K} k$

Note: $\forall \eta \in \mathfrak{X} \otimes k$ gen pt $\forall \mathfrak{X}' \longrightarrow \mathfrak{X}$ abu $\exists! \eta' \longmapsto \eta$ gen pt of strict transform of $\overline{\{\eta\}}$.

Type (5) generalising to $\eta \xleftrightarrow{1:1} C_\eta \setminus \{\eta\}$ where C_η is the irred comp. of η

For $\eta' \longmapsto \eta$, assuming singularities of C_η already resolved, we get $C_{\eta'} \xrightarrow{\cong} C_\eta$ where $C_{\eta'}$ = strict transform

Type (5) = inv limit in these isos

Note. After resolving singularities, we get only strongly semistable models. Additional irred components from blowups are $\mathbb{P}^1$'s.

$\rightarrow$ If $K = \bar{K}$ and X qs + qft / Spa K curve then $\exists$ only fin many type (5) pts $x \in X$ s.t. $\widetilde{\kappa(x)} \not\cong k(t)$

Description of (1), (3), (4) later.

§4 Semistable models of curves

Def. A (strongly) semistable curve is a connected reduced curve / Spec k ... as in §1.

Def. $\mathfrak{X}/\mathcal{O}_K$ is (strongly) semistable model / formal scheme if admissible and $\mathfrak{X} \otimes_{\mathcal{O}_K} k$ is (strongly) sst curve.

Thm. (Sst reduction thm., Artin–Winters, Bosch–Lütkebohmert)

$X^{alg}/\mathrm{Spec}\, K$ smooth proper. Then $\exists L/K$ fin sep and $\mathcal{X}/\mathrm{Spec}\, \mathcal{O}_L$ flat proper model of $X^{alg} \otimes_K L$ s.t. $\mathcal{X} \otimes_{\mathcal{O}_L} k_L$ is strongly sst.

Also holds for X qs qft + smooth curve $/\mathrm{Spa}\, K$.

§5 Skeletons

$\mathbb{D} := \mathbb{D}(0,1)$, $\mathring{\mathbb{D}} = \mathring{\mathbb{D}}(0,1)$

$A(r) := A(\exp(-r), 1)$, $r \in \Gamma$ annulus of ht r

$\mathring{A}(r) := \mathring{A}(\exp(-r), 1)$, $A(\infty) := \mathbb{D} \setminus \{0\} = \bigcup_{r>0} A(r)$, $\mathring{A}(\infty) = \mathring{\mathbb{D}} \setminus \{0\} = \bigcup_{r>0} \mathring{A}(r)$

Def. The skeleton map is
$$\begin{aligned} \sigma : (0,r) &\longrightarrow \mathring{A}(r)^{an} \\ x &\longmapsto \eta_{\mathbb{D}(0,\exp(-r))} \in \mathring{A}(r) \subseteq \mathbb{D} \end{aligned}$$

$\Sigma := \Sigma(\mathring{A}(r)^{an}) := \mathrm{Im}(\sigma)$

Exc. • Height of an annulus is canonical.

• $\Sigma(\mathring{A}(r)^{an})$ is canonical, i.e. any onto $\mathring{A}(r)$ preserves it

• If t is the variable on $A(r)$ then $\mathring{A}(r) \underset{\sigma}{\overset{-\log|t|}{\rightleftarrows}} (0,r)$

In other words, $-\log|t|$ is a retract onto Σ

Prop. 1) σ is an immersion. 2) More precisely if $f \in \mathcal{O}(\mathring{A}(r))$ then $-\log|f| \circ \sigma$ is a piecewise linear function on $(0,r)$.

Pf: 2) $\Rightarrow$ 1): since 1) is a local statement on $A(r)$, wma $r < \infty$. But $A(r)$ is an affinoid adic space, in ptic qc.

$K[t, t^{-1}] \subseteq \mathcal{O}(A(r))$ is dense $\Rightarrow \exists \tilde{f} \in K[t,t^{-1}]$ s.t. $|\tilde{f}| = |f|$ on $\mathring{A}(r)$

By description of the top on $(\mathrm{Spec}\, A)^{an}$, we get 1) from 2).

2): By the same argument, wlog $r < \infty$, $f \in K[t]$, $f = \prod_{i=1}^{n} (t - \alpha_i)$, $\alpha_i \in \overline{K}$.

Then
$$\begin{aligned} |f(\eta_{0,\exp(-r)})| &= \max_{x \in \mathbb{D}(0,\exp(-r))(\overline{K})} |f(x)| \\ &= \exp(-r)^{\#\{i \mid \alpha_i \in \mathbb{D}(0,\exp(-r))\}} \prod_{\alpha_i \notin \mathbb{D}(0,\exp(-r))} |\alpha_i| \end{aligned}$$

Applying $(-\log)$ yields the result. Breakpoints are at $v(\alpha_i)$.

XIV Curves continued

§1 More on models

$K, v, |\cdot|, \mathcal{O}_K, \mathfrak{m}_K$ as usual plus $K = \overline{K}$

Ex. $\mathcal{A}(a) := \operatorname{Spf} \mathcal{O}_K\langle t_1, t_2\rangle / t_1 t_2 - a$ where $a \in \mathfrak{m}_K \setminus \{0\}$

$\mathcal{A}(a)$ is semistable. This is immediate as $\mathcal{A}(a) \otimes k \simeq \operatorname{Spec} k[t_1, t_2]/t_1 t_2$

$\mathcal{A}(a)^{ad}_\eta \simeq A(|a|, 1)$

Equivalent def. $\mathfrak{X}$ is semistable iff $\forall x \in \mathfrak{X}$ $\exists$

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "\text{étale}"'] \arrow[dr, "f", "\text{étale}"'] & \\ \mathcal{A}(a) & & \mathfrak{X} \end{tikzcd} \qquad x \in f(\mathcal{U}).$$

Here by étale we mean: formally étale and lofp on reduced subschemes.

Let $\mathfrak{X}$ be a qcqs sst curve $/\operatorname{Spf} \mathcal{O}_K$, $X := \mathfrak{X}^{ad}_\eta$ qs oft smooth 1-dim $/\operatorname{Spa} K$

Recall that we have

$$\begin{tikzcd} & \mathfrak{X} \\ X^{an} \arrow[ur, "sp^{an}"] \arrow[r, "\text{not cont.}"'] & X \arrow[u, "sp"'] \end{tikzcd}$$

Here sp is continuous and sp^{an} is anti-continuous, i.e. $(sp^{an})^{-1}(\text{open})$ is closed.

Prop. $\xi \in |\mathfrak{X}|$. There are 3 cases:

1) ξ is a generic pt (of an irred component). Then $(sp^{an})^{-1}(\xi)$ is a single type (2) point.
2) ξ is a smooth closed pt. Then $(sp^{an})^{-1}(\xi) \simeq \mathring{\mathbb{D}}^{an}$
3) ξ is a double pt. Then $(sp^{an})^{-1}(\xi) \simeq \mathring{A}(|a|, 1)$ for some $a \in \mathfrak{m}_K \setminus \{0\}$.

Rmk. There is a converse: if $\mathfrak{X}$ is a model for which only these three cases occur then $\mathfrak{X}$ is sst. [Bosch – Lütkebohmert: Unif. of ab. var. I]

Rmk. In case 1), $sp^{-1}(\xi) = \overline{\{(sp^{an})^{-1}(\xi)\}}^{X}$

Pf of Prop: 1): Have $(X, \mathcal{O}_X) \xrightarrow{sp} (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ in LRS. This induces a map in the preimage: $\forall x \in (sp^{an})^{-1}(\xi)$: we get $\mathcal{O}_{\mathfrak{X},\xi}/\mathfrak{m}_\xi = k(\xi) \longrightarrow \widetilde{\kappa(x)}$.

This implies that $\operatorname{trdeg}(\widetilde{\kappa(x)} \mid k) = 1$. Hence x has to be of type (2) by our description of types.

Proof of $\#(sp^{an})^{-1}(\xi)$ is omitted, and not on account of being trivial.

2)+3): $\hat{\mathcal{O}}_{\mathfrak{X},\xi} \simeq \begin{cases} \mathcal{O}_K[[t]] & \text{in case 2)} \\ \hat{\mathcal{O}}_{\mathcal{A}(a),\cdot\cdot} \simeq \mathcal{O}_K[[t_1, t_2]]/t_1 t_2 - a & \text{in case 3) where } a \in \mathfrak{m}_K \setminus \{0\} \end{cases}$

Then $\operatorname{Spf} \hat{\mathcal{O}}_{\mathfrak{X},\eta} \to \mathfrak{X}$, get

$$\begin{tikzcd} \operatorname{Spf} \hat{\mathcal{O}}_{\mathfrak{X},\eta} \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{X} \\ \mathring{\mathbb{D}}^{an} \text{ or } \mathring{A}(|a|,1)^{an} \arrow[u, "sp^{an}"] \arrow[r] & X^{an} \arrow[u, "sp^{an}"'] \end{tikzcd}$$

in pre-adic spaces

RDFINAG, lecture 14

To check ⓐ:

- Can check Zariski-locally on $\mathfrak{X}$. Thus wma $\mathfrak{X} = \mathrm{Spf}\, A$.
- In pre-adic spaces (ref. Scholze–Weinstein, Moduli of p-adic groups) we have an adjunction property for $\mathrm{Spa}(A, A^+)$:

  $\forall (B, B^+)$ sheafy complete Huber pair (note that we need sheafy here)

$$\mathrm{Hom}(\mathrm{Spa}(B,B^+), \mathrm{Spa}(A,A^+)) \xrightarrow{\simeq} \mathrm{Hom}((A,A^+),(B,B^+))$$

- $X^{an} = \{ \mathrm{Im}(\mathrm{Spa}(L, \mathcal{O}_L) \to X) \mid L \text{ non-arch}, \mathcal{O}_L = L^\circ \text{ induces a rk } 1 \text{ valuation} \}$
- sp is functorial, i.e. for $\mathrm{Spa}(L,\mathcal{O}_L) \to X$ we get the following comm. diagram:

$$\begin{tikzcd}
\mathrm{Spf}\, \mathcal{O}_L \arrow[r] & \mathrm{Spf}\, A \\
\mathrm{Spa}(L, \mathcal{O}_L) \arrow[r] \arrow[u, "sp"] \arrow[ur, phantom, "\circlearrowleft"] & (\mathrm{Spf}\, A)^{ad}_\eta \arrow[u, "sp"']
\end{tikzcd}$$

- If $A = \mathcal{O}_K\langle t_1, \dots, t_n \rangle / \mathfrak{a}$ then by the adj-property, $\mathrm{Spa}\, \mathcal{O}_L \to \mathrm{Spa}\, A$ is induced from $A \to \mathcal{O}_L$ sending t_i to $\mathfrak{m}_K$ if $sp(x) = \xi$ where ξ corresponds to $\pi = 0, t_1 = \dots = t_n = 0$.
- Since $\mathfrak{m}_K = \mathcal{O}_L^{\circ\circ}$, the map factors over $\hat{\mathcal{O}}_{L,\xi}$, which yields ⓐ.

Rmk. $sp^{-1}(\xi) \not\cong \mathbb{B}$ or $\mathring{A}(|a|,1) = (\mathrm{Spf}\, \hat{\mathcal{O}}_{\mathfrak{X},\xi})^{ad}_\eta$

§2 Semistable vertex sets

[PBPR], i.e. Baker – Payne – Rabinoff

$X / \mathrm{Spec}\, K$ smooth, irreducible curve, $\hat{X}$ projective compactification

Def. A semistable vertex set for $\hat{X}$ is a finite set $V \subseteq \hat{X}^{an}$ of type (2) points s.t.

$$\hat{X}^{an} \setminus V \simeq \coprod_{\text{inf}} \mathbb{D}^{an} \amalg \coprod_{\text{fin}} \mathring{A}(|a|,1)^{an}.$$

A semistable vertex set for X is a sst vs $V \subseteq \hat{X}^{an}$ for $\hat{X}$ s.t. the fin many $D = \hat{X} \setminus X$ are in distinct $\mathbb{D}^{an}$.

Ex. $\mathfrak{X}/\mathrm{Spf}\, \mathcal{O}_K$ sst model for $\hat{X}$. Then $\hat{X}^{an} = \coprod_{\xi \in \mathfrak{X}} (sp^{an})^{-1}(\xi) \underset{\text{Prop.}}{\simeq} V \amalg \coprod_{\substack{\xi \text{ smooth} \\ \text{closed}}} \mathbb{D}^{an} \amalg \coprod_{\xi \text{ double}} \mathring{A}(|a|,1)^{an}$

where $V := (sp^{an})^{-1}(\text{generic } \xi)$. is a sst vs.

Thm. (BPR, 4.4) $\{\mathfrak{X} \text{ sst model of } \hat{X}\}/_{iso} \simeq \{V \subseteq \hat{X}^{an} \text{ sst vs}\}$

Recall the notion of the skeleton map: $\sigma: (0, v(a)) \longrightarrow \mathring{A}(|a|,1)^{an}$.

Lemma. V sst vs for X, $C \subseteq X^{an} \setminus V$ conn. comp., $\overline{C} :=$ closure in $\hat{X}^{an}$, $\partial C := \overline{C} \setminus C$.

1) If $C \simeq \mathring{\mathbb{D}}^{an}$ then $\partial C = \{pt\}$ is a pt in V.

2) If $C \simeq \mathring{A}(|a|,1)$ then $\sigma : (0, v(a)) \longrightarrow \mathring{A}(|a|,1) \simeq C$ has a unique continuous extension to $[0, v(a)]$ and $\partial C = \{\sigma(0), \sigma(v(a))\}$

3) If $C \simeq \mathring{\mathbb{D}}^{an} \setminus \{0\}$ then $\sigma : (0,\infty) \longrightarrow C$ extends uniquely to $[0,\infty]$ and $\partial C = \{\sigma(0), \sigma(\infty)\}$ and $\sigma(0) \in V$.

Pf: In 1) and 2), $\overline{C} \subseteq X^{an}$ since $C \subseteq \left(\hat{X}^{an} \setminus \bigcup_{d \in D} \mathring{\mathbb{D}}^{an} \setminus \{0\}\right)$.

Since $C \subseteq X^{an} \setminus V$ is closed, $\partial C \subseteq V$.

1): Fix $\varphi : \mathring{\mathbb{D}}^{an} \simeq C$. Choose open affine $\operatorname{Spec} A \simeq X' \subseteq X$ s.t. $C \subseteq (X')^{an}$.

Consider $\sigma : (0,\infty) \longrightarrow C \hookrightarrow (X')^{an}$

$r \longmapsto (f \mapsto \|f\|_r)$

XIII: Given f with fin many zeroes on $A(|a|,1)$, $|f| = \gamma$ pw lin function on $(0, v(a))$ with fin many break points. $\Rightarrow \|f\|_0 := \lim_{r \to 0} \|f\|_r$ gives a semi-norm on A, i.e. $\|\cdot\|_0 \in (X')^{an}$.

Consider $\mathring{\mathbb{D}}^{an} \cup \{\eta_{D(0,1)}\} \subseteq \mathbb{D}^{an}$ with the subspace topology. This is qc.

Define $\mathring{\mathbb{D}}^{an} \cup \{\eta_{D(0,1)}\} \xrightarrow{\tilde{\varphi}} (X')^{an}$, $\tilde{\varphi} = \varphi \amalg (\eta_{(0,1)} \mapsto \|\cdot\|_0)$.

Claim. $\tilde{\varphi}$ cont.

In partic, img in sp is T2 $\Rightarrow$ closed & contains C as a dense open $\Rightarrow \partial C = \|\cdot\|_0$.

PF OF CLAIM: Nts $\forall f \in A$ $\forall (a,b)$: $\tilde{\varphi}^{-1}(|f|^{-1}(a,b))$ is open.

Since $\varphi = \tilde{\varphi}|_C$ is cont, sts continuity at $\eta_{(0,1)}$.

Assume $\|f\|_0 \in (a,b)$. Then choose $\varepsilon > 0$ s.t. f has no zeroes in $\mathring{A}(\varepsilon,1)^{an}$.

Note that $\mathring{A}(\varepsilon,1)^{an} \cup \{\eta_{(0,1)}\}$ is a nbhd of $\eta_{(0,1)}$. Also note that for $0 < r$ small enough, $\|f\|_r \in (a,b)$.

From the computation done in XIII, $x \mapsto \|f\|_{\varphi(x)}$ factors by the retraction to the skeleton

$$\begin{array}{ccc} \mathring{A}(\varepsilon,1)^{an} & \longrightarrow & \Sigma(\mathring{A}(\varepsilon,1)^{an}) \\ & {\scriptstyle |t|} \searrow \quad \swarrow {\scriptstyle \simeq} & \\ & (\varepsilon,1) & \end{array}$$

i.e. $A^\circ(\varepsilon,1) \cup \{\eta_{(0,1)}\} \subseteq \tilde{\varphi}^{-1}(|f|^{-1}(a,b))$ $\square$

Def. V sst vs for X. Then the skeleton of V is

$$\Sigma(X,V) := V \cup \bigcup_{\substack{C \subseteq X^{an} \setminus V \text{ conn comp.} \\ \underbrace{C \simeq A^\circ(|a|,1)^{an}}_{\Sigma \simeq (0, v(a))} \text{ or } \underbrace{\mathring{\mathbb{D}}^{an} \setminus \{0\}}_{\Sigma \simeq (0,\infty)}}} \Sigma(C)$$

- By Lemma: this is a graph wrt subspace topology.
- Canonically metrised from metrics on $(0, v(a))$ and $(0, \infty)$.
- $\Sigma(X, V) \subseteq X^{an}$ is closed, compact iff $X = \hat{X}$ projective
- $\hat{\Sigma}(X,V) := \overline{\Sigma(X,V)}^{\hat{X}^{an}}$ has vertices $V \cup D$.

Def. Strongly sst on V: $\Sigma(X,V)$ has no self-loops.

Prop/Def. $\exists$ cont retraction $\tau_{\Sigma(X,V)} : X^{an} \longrightarrow \Sigma(X,V)$ as follows:

map $C \subseteq X^{an} \setminus \Sigma$ conn. component to $\overline{C} \setminus C$ which is a point in Σ.

Explanation: if $X^{an} \setminus V \simeq \coprod \mathring{\mathbb{D}}^{an} \amalg \coprod \mathring{A}(|a|, 1) \amalg \coprod \mathring{\mathbb{D}}^{an} \setminus \{0\}$ then each $\mathring{\mathbb{D}}^{an}$ is also a conn component of $X^{an} \setminus \Sigma$, τ contracts it to ∂C.

On $\mathring{A}(|a|, 1)^{an}$, $\mathring{\mathbb{D}}^{an} \setminus \{0\}$, τ is contraction onto the skeleton Σ.